ON THE STRUCTURE OF MAXIMALLY ALMOST PERIODIC GROUPS

BY THEODORE W. WILCOX

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1. **Introduction.** A topological group G is said to be maximally almost periodic if the continuous almost periodic functions separate points in G, or equivalently if the continuous finite-dimensional unitary representations of G separate points in G. See [**4**], or [**2**, §18]. Throughout this note, we use "*representation*" to mean "*continuous finite-dimensional unitary representation*". Our purpose here is to announce some results concerning maximally almost periodic (MAP) groups which are independent of the classical theorem of Freudenthal-Weil which states that a locally compact connected group is MAP if and only if it is the direct product of $\boldsymbol{R}^n$ and a compact group [**6**, §§30, 31].

The results in this note comprise a portion of the author's doctoral dissertation. Detailed proofs of these and other results will appear at a later date. The author thanks his thesis advisor, Professor Edwin Hewitt, and Professor Lewis Robertson for all their assistance and encouragement.[1]

2. **Definitions and notation.** Let K be a (Hausdorff but not necessarily locally compact) topological group, G a normal subgroup of K and $T=\{t(x): x\in K\}$ be the group of topological automorphisms of G which are restrictions to G of inner automorphisms of K. Let $\hat{K}$ (and $\hat{G}$ resp.) be the space of equivalence classes of irreducible representations of K (and G resp.). In an investigation of $\hat{K}$ it is natural to consider the action on $\hat{G}$ induced by T. For example, see [**1**]. Let U be a representation, $U\in\sigma\in\hat{G}$, define $t^*(x)U=U\circ t(x)^{-1}$ and define $t^*(x)\sigma$ to be the equivalence class of $t^*(x)U$. If the set $\{t^*(x)\sigma: t(x)\in T\}$ is finite, then σ is said to be *finitely orbited by* T. Let $F(\hat{G},T)$ be the set $\{\sigma\in\hat{G}: \sigma$ is finitely orbited by $T\}$. The *von Neumann kernel* of a group is the intersection of all kernels of representations of that group.

3. **Results.**

THEOREM 1. *Let K, G and T be as above. If $U\in\sigma\in\hat{K}$ and if $y\in G$ are such that $U_y\neq I$, then there exists an element of $F(\hat{G}, T)$ which separates*

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y from the identity. In particular, if K is MAP, then $F(\hat{G}, T)$ separates points in G.

This is proved by utilizing the uniqueness of the decomposition into a direct sum of irreducible constituents of the restriction of U to G; the equivalence classes of these constituents are permuted by the action of T.

THEOREM 2. *Let K, G and T be as above. Let $\sigma \in F(\hat{G}, T)$ and let $O(\sigma, T)$ be the orbit of σ by T. Then the mapping Σ which sends x onto the restriction of $t^*(x)$ to $O(\sigma, T)$ is well defined and is a continuous homomorphism of K onto a finite group. The kernel of Σ contains G.*

In general the condition that $F(\hat{G}, T)$ separate points in G is not enough to imply that K is MAP even if K/G is assumed to be MAP. However, if K is the semidirect product of G and a topological group H, $K = G \circledS_\beta H$, then we have

THEOREM 3. *Let $K = G \circledS_\beta H$. Let H_0 (and $(G \circledS_\beta H)_0$ resp.) be the von Neumann kernel of H (and $G \circledS_\beta H$ resp.). Let $S = \bigcap \{\ker U: U \in \sigma \in F(\hat{G}, \beta(H))\}$. Then $(G \circledS_\beta H)_0 = S \circledS H_0$. In particular, $G \circledS_\beta H$ is MAP if and only if H is MAP and $F(\hat{G}, \beta(H))$ separates points in G.*

The connection between $\beta(H)$ here and the T above follows from the equation $t(e, h)(x, e) = (\beta(h)(x), e)$. See [**2**, p. 7]. The major difficulty in the proof of this theorem is to show that if $g \in G$ and if $U \in \sigma \in F(\hat{G}, \beta(H))$ are such that $U_g \neq I$, then there exists a representation V of K which separates (g, e) from the identity. A rough sketch follows. Let Σ be the homomorphism corresponding to σ defined in Theorem 2. Then $\ker\Sigma = G \circledS M$ and $(G \circledS H)/(G \circledS M)$ is a finite group. Let $\mathfrak{U}(n)$ be the unitary group of U and use Burnside's theorem [**3**, p. 276] to know that the set $\{U_x: x \in G\}$ spans the n^2-dimensional Hilbert space of all linear operators on $\boldsymbol{C}^n$ ($\boldsymbol{C}$ is the field of complex numbers). A closed subgroup $\mathfrak{A}$ of $\mathfrak{U}(n^2)$, a semidirect product $\mathfrak{U}(n) \circledS \mathfrak{A}$ and a continuous homomorphism ϕ of $G \circledS M$ into $\mathfrak{U}(n) \circledS \mathfrak{A}$ are constructed. Then $\phi(g, e)$ can be separated from the identity by a representation W of the compact group $\mathfrak{U}(n) \circledS \mathfrak{A}$ and the desired representation V of K is induced from the representation $W \circ \phi$ of $\ker\Sigma$.

If G is an Abelian group, then we can identify the character group X of G with $\hat{G}$ and with the notation as in 2 above, $F(X, T)$ is a subgroup of X.

THEOREM 4. *Let V be a normal subgroup of a topological group K. Assume further that V is topologically isomorphic to the additive group* *of a finite-dimensional vector space over some locally compact, nondiscrete field of characteristic zero. Let C be the centralizer of V in K. Then K is MAP if and only if C is MAP and K/C is a finite group.*

We make use of Pontrjagin's classification of locally compact fields [**5**, Satz 22] and the fact that the field of real numbers and the p-adic number fields are self-dual to show that the finitely orbited characters of V form a subspace of V, so that $F(V, T)$ is closed in V. Furthermore, it follows from Theorem 1 that $F(V, T)$ is dense in V. These facts imply that T must be finite so that C must have finite index in K.

Using a p-series field, a group can be constructed to show that the hypothesis above (that the field have characteristic zero) is essential.

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University of Washington