GENERALIZATION OF SCHWARZ-PICK LEMMA TO INVARIANT VOLUME IN A KÄHLER MANIFOLD

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Let $\mathfrak{D}$ be the class of bounded homogeneous domains D in the space C^n of n complex variables $z=(z^1, \cdots, z^n)$. A domain D is *homogeneous* if any point of D can be mapped into any other by a holomorphic automorphism. A bounded domain D possesses the Bergman metric, which is invariant under biholomorphic mappings of D, given by

$$ds_D^2 = T_{\alpha\bar{\beta}} dz^\alpha d\bar{z}^\beta \tag{1}$$

(the summation convention is used), where

$$\begin{aligned} T_{\alpha\bar{\beta}} &= T_{\alpha\bar{\beta}}(z, \bar{z}) = (\partial^2 \log K_D)/(\partial z^\alpha \partial \bar{z}^\beta), \\ T_D &= T_D(z, \bar{z}) = \det(T_{\alpha\bar{\beta}}), \end{aligned} \tag{2}$$

and $K_D=K_D(z, \bar{z})$ is the Bergman kernel function of D [2]. The functions $K_D(z, \bar{z})$ and $T_D(z, \bar{z})$ are *relative invariants* of D under biholomorphic mappings and consequently the function

$$I_D(z, \bar{z}) = K_D(z, \bar{z})/T_D(z, \bar{z}) \tag{3}$$

is an *invariant* of D. The kernel function $K_D(z, \bar{z})$ becomes infinite on the boundary of D.

Let $\mathcal{K}$ be the class of Kähler manifolds Δ with metric given by

$$d\sigma_\Delta^2 = g_{\alpha\bar{\beta}}(w, \bar{w}) dw^\alpha d\bar{w}^\beta, \qquad g_\Delta = g_\Delta(w, \bar{w}) = \det(g_{\alpha\bar{\beta}}), \tag{4}$$

where w is a local coordinate of a point on Δ. We also assume

$$-r_{\alpha\bar{\beta}} u^\alpha \bar{u}^\beta \geqq 0 \qquad \text{for any vector } u = (u^\alpha), \tag{5a}$$

$$\det(-r_{\alpha\bar{\beta}}) \geqq g_\Delta, \tag{5b}$$

where $r_{\alpha\bar{\beta}} = -(\partial^2 \log g_\Delta)/(\partial w^\alpha \partial \bar{w}^\beta)$ are the components of the Ricci curvature tensor of the metric (4).

A domain D is *star-like with respect to a point* $z_0 \in D$ if $z \in D$ implies $r(z-z_0) \in D$ for $0<r\leqq 1$. If D is star-like, then the image domains D_r of D under the similarity map

$$w = r(z - z_0), \qquad 0 < r \leqq 1 \tag{6}$$

are such that $D_{r_1} \subset D_{r_2}$ if $r_1 \leqq r_2$, and $D = \bigcup_{j=1}^{\infty} D_{r_j}$, where r_j, $0 < r_j < 1$, is an increasing sequence with limit 1.

THEOREM 1. *If $D \in \mathfrak{D}$ is star-like and can be mapped biholomorphically by $w = w(z)$ into a Kähler manifold $\Delta \in \mathcal{K}$, then*

$$g_\Delta(w, \bar{w}) \mid J_w(z) \mid^2 \leqq T_D(z, \bar{z}) \tag{7}$$

on D, where $J_w(z)$ is the Jacobian of $w = w(z)$.

PROOF (SKETCH). For any $z \in D$, there exists an r such that $z \in D_r$. Let

$$G_{\alpha\bar{\beta}} dz^\alpha d\bar{z}^\beta \tag{8}$$

be the hermitian form on D corresponding to the metric (4) on $\Delta \in \mathcal{K}$ under the inverse mapping $z = z(w)$ of w: $D \rightarrow \Delta$. Then

$$G_D(z, \bar{z}) = \det(G_{\alpha\bar{\beta}}) = g_\Delta(w, \bar{w}) \mid J_w(z) \mid^2 > 0 \tag{9}$$

and

$$\det(-R_{\alpha\bar{\beta}}) \geqq g_\Delta(w, \bar{w}) \mid J_w(z) \mid^2 = G_D(z, \bar{z}),$$
$$R_{\alpha\bar{\beta}} = R_{\alpha\bar{\beta}}(z, \bar{z}) = -(\partial^2 \log G_D)/(\partial z^\alpha \partial \bar{z}^\beta).$$

Let $U = \log (G_D(z, \bar{z})/T_D(0, 0))$, $V = \log(K_{D_r}(z, \bar{z})/K_D(0, 0))$. By a similar argument to that of Dinghas and Ahlfors [1], [3] we show that $U \leqq V$ on D_r, from which Theorem 1 follows.

Let S be a homogeneous Siegel domain of second kind. There exists an increasing sequence $\{S_\nu\}$ of homogeneous subdomains with limit S and such that $S_\nu \cup \partial S_\nu \subset S_{\nu+1}$, where ∂S_ν is the boundary of S_ν lying in (finite) C^n space. Let $\mathcal{K}'$ be that subclass of $\mathcal{K}$ for which the metric can be chosen so that

$$\lim_{\zeta \rightarrow \zeta_\infty} G_S(\zeta, \bar{\zeta})/T_{S_\nu}(\zeta, \bar{\zeta}) \leqq 1, \qquad \zeta \in S_\nu, \tag{10}$$

for all ν sufficiently large where ζ_∞ is a boundary point of S_ν which is "a point at infinity." The class $\mathcal{K}'$ is nonempty. An extension of Theorem 1 to domains S is given in

THEOREM 2. *Let S be a homogeneous Siegel domain of second kind. If $w = w(\zeta)$ maps S biholomorphically into a Kähler manifold $\Delta \in \mathcal{K}'$, then $g_\Delta(w, \bar{w}) \mid J_w(\zeta) \mid^2 \leqq T_S(\zeta, \bar{\zeta})$ on S.*

Then by a well-known result of Vinberg, Gindikin and Pjateckiĭ-Šapiro [7] that every bounded homogeneous domain D can be mapped biholomorphically onto an affinely homogeneous Siegel domain of second kind we have

THEOREM 3. *If $D \in \mathfrak{D}$ can be mapped into a Kähler manifold Δ in $\mathcal{K}'$ by a biholomorphic mapping $w = w(z)$, then for $z \in D$,*

$$g_D(w, \bar{w}) \mid J_w(z) \mid^2 \leqq T_D(z, \bar{z}).$$

PROOF OF THEOREM 2 (SKETCH). By the result of Pjateckiĭ-Šapiro [6] that the Siegel domain of second kind is biholomorphically equivalent to a bounded domain D we get an increasing sequence of homogeneous bounded subdomains D_ν, with union D, corresponding to the sequence $\{S_\nu\}$ for S. Then an analogous argument to that in Theorem 1 is applicable.

REMARK. Full details will appear in [4] and [5].

REFERENCES

1. L. V. Ahlfors, *An extension of Schwarz's lemma*, Trans. Amer. Math. Soc. **43** (1938), 359–364.

2. S. Bergman, *Sur les fonctions orthogonales de plusiers variables complexes avec les applications à la theorie des fonctions analytiques*, Mem. Sci. Math. **106** (1947), Paris.

3. A. Dinghas, *"Ein N-dimensionalen Analogen des Schwarz-Pickschen Flächensatzes für holomorphe Abbildungen der komplexen Einheitskugel in eine Kähler Mannigfaltigkeit"* in *Festschrift zur Gedachtnisfeier für Karl Weierstrass* 1815–1965, Wiss. Abh. Arbeitsgemeinschaft Forsch. des Landes Nordr.-Westfalen, Bd. 33, Westdeutscher Verlag, Cologne, 1966.

4. K. T. Hahn and Josephine Mitchell, *Generalization of Schwarz-Pick lemma to invariant volume in Kähler manifold*, Trans Amer. Math. Soc. **127** (1967).

5. ———, *Generalization of Schwarz-Pick lemma to invariant volume in a Kähler manifold*. II, (to appear).

6. I. I. Pjateckiĭ-Šapiro, *Geometry of classical domains and theory of automorphic functions*, Fizmatgiz, Moscow, 1961.

7. È. B. Vinberg, S. G. Gindikin and I. I. Pjateckiĭ-Šapiro, *Classification and canonical realization of complex homogeneous bounded domains*, Trudy Moscow, Mat. Obsc. **12** (1963), 359–388, Trans. Moscow Math. Soc. **12** (1963), 404–437.

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