PIECEWISE MONOTONE POLYNOMIAL INTERPOLATION

BY SAM W. YOUNG

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The purpose of this paper is to prove the following:

THEOREM. *Suppose that n is a positive integer, $x_{i-1}<x_i$ and $y_{i-1}\neq y_i$, $i=1, 2, \cdots, n$. Then there exists a polynomial P such that $P(x_i)=y_i$, $i=0, 1, \cdots, n$ and P is monotone in each of the intervals $[x_{i-1}, x_i]$, $i=1, 2, \cdots, n$.*

This theorem differs from the usual polynomial interpolation theorem in that there is no mention of the degree of the polynomial. The proof presented here leaves little room for generalization from ordinary polynomials to more general systems of functions since it is essential that zeros can be prescribed.

PROOF. Let $y_0=0$ and generality is not lost. $D=\{Q$: Q is a polynomial and $Q(x)(y_i-y_{i-1})\geqq 0$ for $x_{i-1}\leqq x\leqq x_i$, $i=1, 2, \cdots, n\}$. If $Q_1,Q_2\in D$ and $0\leqq t\leqq 1$, then $[tQ_1(x)+(1-t)Q_2(x)]\ (y_i-y_{i-1}) =tQ_1(x)(y_i-y_{i-1})+(1-t)Q_2(x)(y_i-y_{i-1})\geqq 0$ for $x_{i-1}\leqq x\leqq x_i$ and $i=1, 2, \cdots, n$. Therefore D is a convex subset of the space $C[x_0, x_n]$. Furthermore, if $Q\in D$ and $a\geqq 0$, then $aQ\in D$.

Let F be the function from D into E_n defined by

$$F(Q) = \left\{\int_{x_0}^{x_i} Q(x)dx\right\}_{i=1}^{n}.$$

F is linear and so $F(D)$ is a convex subset of E_n. Furthermore if $z\in F(D)$ and $a\geqq 0$, then $az\in F(D)$.

For each i, $1\leqq i\leqq n$, let ϕ_i denote the point of E_n such that if $1\leqq j<i$, then the jth coordinate is zero and if $i\leqq j\leqq n$, then the jth coordinate is one. Let $\lambda_i=\text{sign }(y_i-y_{i-1})\cdot\phi_i$ and note that $\phi_i=\text{sign }(y_i-y_{i-1})\cdot\lambda_i$. Any point $Z=(z_1, z_2, \cdots, z_n)$ has the representation $z=\sum (z_i-z_{i-1})\phi_i$ where $z_0=0$. In particular, if $Q\in D$,

$$F(Q) = \sum\left(\int_{x_{i-1}}^{x_i} Q(x)dx\right)\cdot\phi_i = \sum\left(\int_{x_{i-1}}^{x_i} Q(x)dx\right)\text{sign}(y_i - y_{i-1})\cdot\lambda_i.$$

But

$$\text{sign}\left(\int_{x_{i-1}}^{x_i} Q(x)dx\right) = \text{sign}(y_i - y_{i-1})$$

since $Q(x)[y_i - y_{i-1}] \geqq 0$ for all $x_{i-1} \leqq x \leqq x_i$. Therefore we have the equation

$$F(Q) = \sum \left| \int_{x_{i-1}}^{x_i} Q(x)dx \right| \cdot \lambda_i.$$

If $Q \in D$ and $Q \not\equiv 0$ and we multiply Q by a nonnegative polynomial P which is near $\left| \int_{x_{i-1}}^{x_i} Q(x)dx \right|^{-1}$ between x_{i-1} and x_i and which is small elsewhere, then the product $QP \in D$ and $F(QP)$ is near λ_i. Thus each of the points λ_i is a limit point of $F(D)$. An exact verification of this assertion requires a lengthy but unsophisticated calculation.

Let $y = (y_1, y_2, \cdots, y_n)$. We must now show that $y \in F(D)$. The points $\{\lambda_i\}$ form a basis for E_n and $y = \sum (y_i - y_{i-1}) \cdot \phi_i = \sum (y_i - y_{i-1}) \cdot \text{sign } (y_i - y_{i-1}) \cdot \lambda_i = \sum |y_i - y_{i-1}| \cdot \lambda_i$ where by hypothesis, $|y_i - y_{i-1}| > 0$, $i = 1, 2, \cdots, n$. Since $\{\lambda_i\}$ is a basis each member of which is a limit point of $F(D)$, we can choose a basis $\{\theta_i\}$ from the set $F(D)$ close enough to $\{\lambda_i\}$ that if $y = \sum t_i \theta_i$, then $t_i > 0$, $i = 1, 2, \cdots, n$. Let $\sigma = \sum t_i$. By convexity of $F(D)$, $\sigma^{-1} y = \sigma^{-1} \sum t_i \theta_i$ belongs to $F(D)$ and therefore $\sigma\sigma^{-1} y = y \in F(D)$.

Since $y \in F(D)$, there exists a $Q \in D$ such that $F(Q) = y$. Thus if $P(t) = \int_{x_0}^{t} Q(x)dx$, then $P(t)$ is a polynomial which satisfies the conclusion of the theorem.

We have used the Weierstrass theorem in the proof of this theorem. The Weierstrass theorem is also an immediate consequence of this interpolation theorem and so the two are equivalent.

UNIVERSITY OF UTAH