RESTRICTED REPRESENTATIONS OF CLASSICAL LIE ALGEBRAS OF TYPES A_2 AND B_2[1]

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The dimensions of the finite dimensional irreducible restricted modules for a Lie algebra of classical type have never been determined. C. W. Curtis ([5], [6]) has given sufficient, but not necessary, conditions that the dimension be given by Weyl's formula. In this paper, we give results which determine the dimensions of a certain class of finite dimensional irreducible restricted modules for a simple algebra of type A_2 or B_2 over a field of characteristic $p>3$. Counterexamples to a conjecture of N. Jacobson regarding the complete reducibility of certain modules for an algebra of classical type are given. Our method involves rather lengthy, though elementary, calculations (given in detail in [3]), and depends on a character formula for algebras of types A_2 or B_2 over the complex field, which is due to J. P. Antoine ([1], [2]). R. Steinberg has mentioned that the results for A_2 were obtained by Mark ([9]).

1. **Preliminaries.** Let $\mathfrak{L}$ be a simple Lie algebra over the complex field with Cartan subalgebra $\mathfrak{H}$. If $p>3$ is a prime number, let $\bar{\mathfrak{L}}$, $\bar{\mathfrak{H}}$ be the classical Lie algebra and corresponding Cartan subalgebra over the field $Z/(p)$ obtained from $\mathfrak{L}$ and $\mathfrak{H}$ by reducing the structure constants with respect to a Chevalley basis, modulo p (see e.g., [10]). The finite dimensional irreducible restricted $\bar{\mathfrak{L}}$-modules were shown by Curtis to be uniquely determined by their maximal weights [4]. Given a linear function $\bar{\Lambda}$ on $\bar{\mathfrak{H}}$, let Λ be the integral linear function on $\mathfrak{H}$ such that for $i=1, 2, \cdots l$, $\Lambda(h_i)$ is the integer between 0 and $p-1$ whose residue modulo p is $\bar{\Lambda}(h_i)$. (Here $h_1, \cdots, h_l$ is a basis of co-roots for $\mathfrak{H}$, corresponding to a system of fundamental roots $\alpha_1, \alpha_2, \cdots, \alpha_l$, and we have identified these elements of $\mathfrak{H}$ with their counterparts in $\bar{\mathfrak{H}}$.) Then Λ is a dominant integral function on $\mathfrak{H}$, and hence is the highest weight of a unique finite dimensional irreducible $\mathfrak{L}$-module V. Now let $\bar{V}$ be the (not necessarily restricted) $\bar{\mathfrak{L}}$-module corresponding to V (see [5]). Curtis showed ([5]) that the finite dimensional irreducible restricted $\bar{\mathfrak{L}}$-module with maximal

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weight $\bar{\Lambda}$ is a homomorphic image of $\bar{V}$. In this direction we have the following results:

THEOREM 1. *If $\bar{V}$ is restricted, then it is indecomposable and has a unique maximal submodule.*

THEOREM 2. *If $\Lambda(h_\alpha) < p$ for all roots α of $\mathfrak{L}$ with respect to $\mathfrak{H}$, then $\bar{V}$ is restricted. (For algebras of type A_2 this condition is $\Lambda(h_1)+\Lambda(h_2) < p$, and for $\mathfrak{L}$ of type B_2 it becomes $2\Lambda(h_1)+\Lambda(h_2) < p$.)*

2. **Restricted modules for algebras of type A_2.** The simple Lie algebra $\mathfrak{L}$ of type A_2 over the complex field C has a Chevalley basis $\{e_1, e_2, e_3, f_1, f_2, f_3, h_1, h_2\}$ with $[e_3f_1] = -e_2$, $[e_3f_2] = e_1$, $[f_3e_1] = -f_2$, $[f_3e_2] = f_1$, such that $\mathfrak{H} = Ch_1 + Ch_2$ is a Cartan subalgebra and if $\alpha_3 = \alpha_1 + \alpha_2$ (the sum of the fundamental roots) and $h_3 = h_1 + h_2$, then $\{e_i, f_i, h_i\}$ is a standard basis for the three dimensional simple subalgebra of $\mathfrak{L}$ with roots $\pm\alpha_i$, $i = 1, 2, 3$. The representation theory of the subalgebras $\mathfrak{L}^i = \mathfrak{H} + Ce_i + Cf_i$ is known [7, p. 113], and we use it to study the finite dimensional irreducible $\mathfrak{L}$-modules.

Given nonnegative integers r and s, let $V_{r,s}$ be the finite dimensional irreducible $\mathfrak{L}$-module with highest weight Λ satisfying $\Lambda(h_1) = r$, $\Lambda(h_2) = s$. Any irreducible $\mathfrak{L}^i$-submodule of $V_{r,s}$ is generated by an e_i*-extreme* vector, i.e., a nonzero weight vector which is annihilated by e_i. We determine the decomposition of $V_{r,s}$ into a sum of irreducible $\mathfrak{L}^3$-submodules by finding all e_3-extreme vectors.

The Λ-weight space of $V_{r,s}$ is one dimensional and is annihilated by the positive root elements e_i, $i = 1, 2, 3$. If v is any nonzero element of this weight space, set $X_{r-j,0} = vf_1^{r-j}$, $j = 0, 1, \cdots, r$, and for $k \geqq 1$, define $X_{r-j,k}$ inductively by

$$X_{r-j,k} = (j - k + 1)(r + j - k)X_{r-j,k-1}f_3 + (s + j - 2k + 2)X_{r-j+1,k-1}f_2,$$

$j = k, k+1, \cdots, r$. Then these vectors are e_3-extreme in $V_{r,s}$, $X_{r-j,k}$ having weight $\Lambda - (r-j)\alpha_1 - k\alpha_3$. Similarly, $Y_{s-j,0} = vf_2^{s-j}$, $j = 0, 1, \cdots, s$, and for $k \geqq 1$,

$$Y_{s-j,k} = (j - k + 1)(s - j + k)Y_{s-j,k-1}f_3 - (r + j - 2k + 2)Y_{s-j+1,k-1}f_1,$$

$j = k, k+1, \cdots, s$, are e_3-extreme vectors of weight $\Lambda - (s-j)\alpha_2 - k\alpha_3$. One can show that if $r-k$ and $s-k$ are nonnegative, then $X_{r-j,k}$ and $Y_{s-j,k}$ are nonzero, and $Y_{0,k} = (-1)^k X_{0,k}$. Then, using Antoine's character formula, which gives the weight space decomposition of

$V_{r,s}$ explicitly, it can be seen that the vectors $X_{r,j,k}f_3^m$, $m=0, 1, \cdots, s-j+2k$, and $Y_{s-j,k}f_3^m$, $m=0, 1, \cdots, r+j-2k$, form a basis for $V_{r,s}$. Moreover, consideration of the effect of the root elements of $\mathfrak{L}$ on these vectors shows that they may be taken as a basis for the $\overline{\mathfrak{L}}$-module $\overline{V}_{r,s}$ provided $r+s<p$. (Note that this is precisely the condition in Theorem 2 which guarantees that $\overline{V}_{r,s}$ is restricted.) Using this basis, we obtain the following results:

THEOREM 3. *If $r+s<p-1$, then $\overline{V}_{r,s}$ is irreducible.*

THEOREM 4. *$\overline{V}_{p-1,0}$ and $\overline{V}_{0,p-1}$ are irreducible. However, if $r>0$, $s>0$ and $r+s=p-1$, then $\overline{V}_{r,s}$ is reducible. Its unique maximal submodule is generated by the vector $X_{0,1}$, and is isomorphic to $\overline{V}_{r-1,s-1}$.*

COROLLARY. *If $\overline{\mathfrak{L}}$ is a simple Lie algebra of type A_2 over a field of characteristic $p>3$, let $M_{r,s}$ be the irreducible restricted $\overline{\mathfrak{L}}$-module with maximal weight $\overline{\Lambda}=(\bar{r}, \bar{s})$, where $\bar{r}$, $\bar{s}$ are the residues of the nonnegative integers r and s, modulo p. Then*

(i) if $r+s<p-1$, dim $M_{r,s}=\frac{1}{2}(r+1)(s+1)(r+s+2)$,
(ii) dim $M_{p-1,0}=$dim $M_{0,p-1}=\frac{1}{2}p_1p+1)$,
(iii) if $r>0$, $s>0$ and $r+s=p-1$,

$$\dim M_{r,s} = \tfrac{1}{2}p(p+1) + rs.$$

PROOF. The modules $\overline{V}_{r,s}$, $r+s<p$, are all restricted, by Theorem 2. In cases (i) and (ii), we have shown that $\overline{V}_{r,s}$ is irreducible, so since both $M_{r,s}$ and $\overline{V}_{r,s}$ are irreducible restricted $\overline{\mathfrak{L}}$-modules with the same highest weight, Curtis's classification theorem ([**4**]) shows that they are isomorphic. Thus dim $M_{r,s}=$dim $\overline{V}_{r,s}=$dim $V_{r,s}$, which is given by Weyl's formula ([**7**, p. 257]): dim $V_{r,s}=\frac{1}{2}(r+1)(s+1)(r+s+2)$.

In case (iii), we have shown that the unique maximal submodule W of $\overline{V}_{r,s}$ is isomorphic to $\overline{V}_{r-1,s-1}$. Since ([**5**]) $M_{r,s}$ is a homomorphic image of $\overline{V}_{r,s}$, we conclude that $M_{r,s}\cong\overline{V}_{r,s}/W$ and

$$\begin{aligned}\dim M_{r,s} &= \dim V_{r,s} - \dim V_{r-1,s-1} \\ &= \tfrac{1}{2}(r+1)(s+1)(r+s+2) - \tfrac{1}{2}rs(p-1) \\ &= \tfrac{1}{2}(rs+p)(p+1) - \tfrac{1}{2}rs(p-1) = \tfrac{1}{2}p(p+1) + rs.\end{aligned}$$

3. **Restricted modules for algebras of type B_2.** The simple algebra of type B_2 has fundamental roots α_1, α_2 and two composite roots, $\alpha_3=\alpha_1+\alpha_2$, $\alpha_4=\alpha_1+2\alpha_2$. The corresponding co-roots are h_1, h_2, $h_3=2h_1+h_2$ and $h_4=h_1+h_2$. If $V_{r,s}$ is defined just as before, the decomposition of $V_{r,s}$ as an $\mathfrak{L}^1$ and $\mathfrak{L}^4$-module can be determined. Thus one can find vectors $Z_{s-j,k}$ of weight $\Lambda-(s-j)\alpha_2-k\alpha_3$, $j=0, 1, \cdots, s$, $k=0, 1, \cdots, r$, which are simultaneously e_1 and e_4-extreme, such

that the nonzero vectors $Z_{s-j,k}f_1^m f_4^n$ are a basis for $V_{r,s}$. Then, examining the effect of the basis elements of $\mathfrak{L}$ on these vectors, one shows that they can serve as a basis for $\overline{V}_{r,s}$ precisely when $2r+s<p$. The computations are lengthy, and we refer to [3] for details. The upshot of this investigation is summarized in Theorems 5, 6, and 7.

THEOREM 5. *If* $2r+s<p-2$, $\overline{V}_{r,s}$ *is irreducible.*

THEOREM 6. $\overline{V}_{0,p-2}$ *is irreducible. However, if* $r>0$ *and* $2r+s=p-2$, *then* $\overline{V}_{r,s}$ *is reducible. Its unique maximal submodule is generated by* $Z_{0,1}$, *and is isomorphic to* $\overline{V}_{r-1,s}$.

REMARK. The modules $\overline{V}_{r,s}$, $r>0$, $s>0$, $2r+s=p-2$, dealt with in Theorem 6 give counter-examples to the following conjecture of Jacobson ([8, p. 830]): Complete reducibility holds for any representation R of a Lie algebra $\overline{\mathfrak{L}}$ of classical type over a field of characteristic p in which $(e_\alpha^R)^{p-1}=0$ for all roots $\alpha\neq 0$ of $\overline{\mathfrak{L}}$. According to Theorem 2, the modules $\overline{V}_{r,s}$ are restricted, so by Theorem 1 they are indecomposable. But, by Theorem 6, they are not irreducible and hence not completely reducible. By examining the weight diagrams, one sees that $e_\alpha^{p-1}=0$ on the modules $V_{r,s}$ for all Chevalley root elements of $\mathfrak{L}$, and hence $e_\alpha^{p-1}=0$ on the $\overline{L}$-modules $\overline{V}_{r,s}$.

THEOREM 7. $\overline{V}_{0,p-1}$ *and* $\overline{V}_{1,p-3}$ *are irreducible. However, if* $r>1$ *and* $2r+s=p-1$, *then* $\overline{V}_{r,s}$ *is reducible. Its unique maximal submodule is generated by* $Z_{0,2}$, *and is isomorphic to* $\overline{V}_{r-2,s}$.

Combining Theorems 5–7 with Weyl's formula for the dimensions of the modules $V_{r,s}$ for $\mathfrak{L}$ of type B_2

$$(\dim V_{r,s} = (1/6)(r+1)(s+1)\cdot(2r+s+3)(r+s+2)),$$

one can determine the dimensions of the irreducible restricted $\mathfrak{L}$-modules $M_{r,s}$ for $2r+s<p$, as was done for A_2 in the Corollary following Theorem 4.

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