STRUCTURE OF A CLASS OF REGULAR SEMIGROUPS AND RINGS

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One of the most natural approaches to the study of regular semigroups is to impose restrictions on the partial ordering of their idempotents ($e \leqq f \Leftrightarrow ef = fe = e$). The principal object of this note is to describe the structure of the classes of regular semigroups whose idempotents form a tree or a unitary tree, respectively (see Definition 1). We determine, among other things, a complete set of invariants, isomorphisms, the group of automorphisms, and congruences of these semigroups. We also consider regular rings whose multiplicative semigroup satisfies conditions (C) or (C_1) and give their structure. The terminology concerning semigroups is that of [2] and concerning p.o. sets of [1]. We consider only semigroups with zero; the statements concerning semigroups without zero can then be easily deduced.

DEFINITION 1. A [*unitary*] *tree* T is a p.o. set with a unique minimal element 0 [and a unique maximal element 1], $T \neq \{0\}$, satisfying

(i) all elements [different from 1] are of finite height;

(ii) every element different from 0 [and different from 1] covers exactly one element.

DEFINITION 2. A regular semigroup whose p.o. set of idempotents is a tree [unitary tree] is said to be $\mathfrak{J}$-*regular* [$\mathfrak{J}_1$-*regular*].

A $\mathfrak{J}_1$-regular semigroup has an identity element. In order to find the structure of such semigroups, we need the following construction. For any semigroup S with zero, we write $S^* = S \backslash 0$.

Let T be a tree; to every $\alpha \in T^* = T \backslash 0$ associate a semigroup S_α with zero 0_α; the semigroups S_α are pairwise disjoint. If $\alpha\chi > 1$ ($\alpha\chi$ is the height of α in T), associate to α a partial homomorphism ϕ_α: $S_\alpha^* \rightarrow S_{\bar{\alpha}}^*$ ($\bar{\alpha}$ is the unique element of T covered by α). On the set $V = (\bigcup_{\alpha \in T^*} S_\alpha^*) \cup 0$, multiplication is defined by induction on the height of $\alpha \in T$ as follows. Let 0 act as the zero of V. If $\alpha\chi = \beta\chi = 1$ and $x \in S_\alpha^*$, $y \in S_\beta^*$ (multiplication in S_α is denoted by juxtaposition), let

$$x \circ y = xy \qquad \text{if } \alpha = \beta,\ xy \neq 0_\alpha,$$
$$= 0 \qquad \text{if } \alpha = \beta,\ xy = 0_\alpha \text{ or } \alpha \neq \beta.$$

Supposing that multiplication has been defined for all $u \in S_\gamma^*$, $v \in S_\delta^*$, $\gamma\chi$, $\delta\chi < n$ ($n > 1$), for $x \in S_\alpha^*$, $y \in S_\beta^*$ with $\alpha\chi$, $\beta\chi \leqq n$, let

$$
\begin{aligned}
x \circ y &= xy, \quad \text{if } \alpha = \beta,\ xy \neq 0_\alpha,\ \alpha\chi = n,\\
&= x\phi_\alpha \circ y\phi_\beta, \quad \text{if } \alpha = \beta,\ xy = 0_\alpha \text{ or } \alpha \neq \beta,\ \alpha\chi = \beta\chi = n,\\
&= x\phi_\alpha \circ y, \quad \text{if } \alpha\chi = n,\ \beta\chi < n,\\
&= x \circ y\phi_\beta, \quad \text{if } \alpha\chi < n,\ \beta\chi = n.
\end{aligned}
$$

Under this multiplication, V is a semigroup called a *tree* of semigroups S_α with *support* T; we write $V=(T;\ S_\alpha,\ \phi_\alpha)$. Using ideal extensions, a proof by induction on the height of idempotents establishes the following theorem, which is fundamental for our study.

THEOREM 1. *A semigroup V is $\mathfrak{J}$-regular if and only if V is a tree of completely* 0-*simple semigroups. If $V=(T;\ S_\alpha,\ \phi_\alpha)$, then the set $\{T,\ S_\alpha,\ \phi_\alpha\}$ constitutes a complete set of invariants of V (i.e., if also $V'=(T';\ S_{\alpha'},\ \phi_{\alpha'})$, then $V\cong V'$ if and only if* (a) *there is an order isomorphism ξ of T onto T',* (b) *for every $\alpha\in T^*$, there is an isomorphism η_α of S_α onto $S'_{\alpha\xi}$ such that if $\alpha\chi>1$, then for every $x\in S^*_\alpha$, $x\phi_\alpha\eta_{\bar\alpha}=x\eta_\alpha\phi'_{\alpha\xi}$).*

Let $S=(T_0;\ S_\alpha,\ \phi_\alpha)$ be a $\mathfrak{J}$-regular semigroup and let G be a group. For each $\alpha\in T_0^*$, let Φ_α be a homomorphism of G into the group U_α of units of the translational hull $\Omega(S_\alpha)$ of S_α; write $g\Phi_\alpha=(\lambda^g_\alpha,\ \rho^g_\alpha)$ (pair of linked left and right translations of S_α, where λ^g_α and ρ^g_α are written as operators on the left and right, respectively). Suppose that the diagram

$$
\begin{array}{ccccc}
S^*_\alpha & \xrightarrow{\ \lambda^g_\alpha\ } & S^*_\alpha & \xleftarrow{\ \rho^g_\alpha\ } & S^*_\alpha\\
\phi_\alpha\big\downarrow & & \phi_\alpha\big\downarrow & & \phi_\alpha\big\downarrow\\
S^*_{\bar\alpha} & \xrightarrow{\ \lambda^g_{\bar\alpha}\ } & S^*_{\bar\alpha} & \xleftarrow{\ \rho^g_{\bar\alpha}\ } & S^*_{\bar\alpha}
\end{array}
$$

commutes for every $g\in G$, $\alpha\chi>1$. Denoting multiplication in S and G by juxtaposition, define multiplication on the set $\Sigma=S\cup G$ by

$$
\begin{aligned}
x \circ y &= xy, \quad \text{if } x, y \in S \text{ or } x, y \in G,\\
&= x\overset{y}{\rho}_\alpha, \quad \text{if } x \in S^*_\alpha,\ y \in G,\\
&= \overset{x}{\lambda}_\alpha y, \quad \text{if } x \in G,\ y \in S^*_\alpha,\\
&= 0, \quad \text{if } x = 0 \text{ or } y = 0.
\end{aligned}
$$

Write $\Sigma=(T;\ S_\alpha,\ \phi_\alpha;\ \Phi_\alpha)$ where $1\notin T_0$, $T=T_0\cup 1$, $S_1=G$. Letting $1>\alpha$ for all $\alpha\in T_0$, T becomes a unitary tree.

THEOREM 2. *Under this multiplication, Σ is a $\mathfrak{J}_1$-regular semigroup and the set $\{T,\ S_\alpha,\ \phi_\alpha,\ \Phi_\alpha\}$ constitutes a complete set of invariants of Σ. Conversely, every $\mathfrak{J}_1$-regular semigroup can be constructed in this fashion.*

Isomorphisms between two $\mathfrak{J}_1$-regular semigroups can be conveniently expressed using matrices. If $\Sigma=(T;\ S_\alpha,\ \phi_\alpha;\ \Phi_\alpha)$ is $\mathfrak{J}_1$-regular, let M be the $T^*\times T^*$-matrix, $M=(m_{\alpha\beta})$ where

$$\begin{aligned} m_{\alpha\beta} &= \phi_\alpha, && \text{if } \beta = \bar{\alpha},\ \alpha \neq 1,\ \alpha\chi > 1, \\ &= \Phi_\beta, && \text{if } \alpha = 1,\ \beta \neq 1, \\ &= 0, && \text{otherwise.} \end{aligned}$$

The semigroup Σ is completely determined by semigroups S_α and the matrix M; we write $\Sigma=(M;\ S_\alpha)$.

COROLLARY. *In order that the* $\mathfrak{J}_1$*-regular semigroups* $\Sigma=(M;\ S_\alpha)$ *and* $\Sigma'=(M';\ S'_{\alpha'})$ *be isomorphic it is necessary and sufficient that there exists an invertible matrix* A *whose nonzero elements are isomorphisms of* $\Omega(S_\alpha)$ *onto* $\Omega(S'_{\beta'})$ *and such that* $MA=AM'$ ($\Omega(S)$ *is the translational hull of* S).

This corollary yields also the group of automorphisms of a $\mathfrak{J}_1$-regular semigroup $\Sigma=(M;\ S_\alpha)$. In fact, the group of automorphisms of Σ is isomorphic to the group of matrices A described in the corollary with $MA=AM$.

$\mathfrak{J}$-regular semigroups can also be characterized in terms of ideals and subdirect products as follows.

THEOREM 3. *A semigroup* S *is* $\mathfrak{J}$*-regular if and only if* S *is regular, has a zero,* $S\neq(0)$, *and satisfies*:

(i) *every principal two-sided ideal* I *of* S *is of finite height (in the p. o. set of two-sided ideals of* S*) and is quasicompletely prime (i.e.,* $xyz\in I$ *implies* $xy\in I$ *or* $yz\in I$*; see* [**3**]);

(ii) S *is a subdirect product of completely* 0*-simple semigroups.*

A nonzero homomorphic image of a $\mathfrak{J}_1$-regular semigroup is again $\mathfrak{J}_1$-regular. Homomorphisms of a $\mathfrak{J}$-regular semigroup onto another can be found using homomorphisms of its support (which is a semilattice) and of different S_α; these homomorphisms must satisfy certain compatibility conditions. Using this and a characterization of congruences σ on a $\mathfrak{J}_1$-regular semigroup S such that S/σ is a group with zero, congruences on a $\mathfrak{J}_1$-regular $\Sigma=(T;\ S_\alpha,\ \phi_\alpha;\ \Phi_\alpha)$ can be expressed by means of an arbitrary subset X of T containing its minimal element 0, congruences on S_α for $\alpha\neq 0$, $\alpha\in X$, satisfying some compatibility conditions.

All the statements above have their analogues for $\mathfrak{J}$-regular semigroups.

When considering rings whose multiplicative semigroup is $\mathfrak{J}$- or $\mathfrak{J}_1$-regular, we obtain the structure of a somewhat larger class of regular rings. On a ring R consider the conditions:

(C) for all idempotents e, f, $g \in R$, $e<f$, $g<f$ implies e and g are comparable;

(C_1) R has a unit 1 and for all idempotents e, f, $g \in R$, $e<f$, $g<f$ implies e and g are comparable or $f=1$.

THEOREM 4. *Let R be a regular ring, $R \neq (0)$. Then*

(i) *R satisfies* (C) *if and only if R is a division ring;*

(ii) *R satisfies* (C_1) *if and only if R is either a division ring, or a direct sum of two division rings, or the ring of 2×2 matrices over a division ring.*

Examples of the semigroups studied can be easily found among different semigroups of (partial) transformations on a set, semigroups of endomorphisms of a linear manifold, etc. A (very) special case of a $\mathfrak{I}$-regular semigroup is an orthogonal sum of completely 0-simple semigroups [**3**].

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