ON THE CLOSURE OF CERTAIN BANACH SPACES OF FUNCTIONS OF SEVERAL VARIABLES

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1. **Statement of the main results.** Our primary goal in this note is to establish the following proposition.

THEOREM 1. *Consider the Banach space* $\mathfrak{C}_n$ *of continuous real-valued functions* $f: E_n \to R$, E_n *standing for the unit cube in n-dimensional Euclidean space. If* ϕ *and* ψ *are any fixed functions of* $\mathfrak{C}_n$ *with connected level-sets intersecting pairwise in connected sets, then the subspace of superpositions* $a \circ \phi + b \circ \psi$ *is closed in* $\mathfrak{C}_n$ *under the uniform norm.*

Mark by $\mathfrak{G}_n$ the indicated space of superpositions. To prove the stated theorem, it suffices to verify

THEOREM 2. *Every function of* $\mathfrak{C}_n$ *has a best uniform approximation in* $\mathfrak{G}_n$.

Distinguish one of the fixed functions, say, ψ; denote its level sets by $l_\psi(t)$,

$$l_\psi(t) = \left\{ p \in E_n : \psi(p) = t \right\};$$

designate by L_ψ the aggregate of level sets $l_\psi = l_\psi(t)$. Finally, set for each $f \in \mathfrak{C}_n$

$$\omega(f \mid l_\psi) = \max_{p \in l} f(p) - \min_{p \in l_\psi} f(p),$$

$$\omega(f \mid \psi) = \max_{l_\psi \in L_\psi} \omega(f \mid l_\psi),$$

$$\mu(f) = \inf_{\mathfrak{G}_n} \|f - a \circ \phi - b \circ \psi\|,$$

(the properties of the functional ω and related topics are investigated in [1] and [2]. Theorem 2 is proved by means of the five lemmas now formulated.

LEMMA 1. *For each* $f \in \mathfrak{C}_n$,

$$\mu(f) = \tfrac{1}{2} \inf_{a \in \mathfrak{C}} \omega(f - a \circ \phi \mid \psi),$$

where $\mathbb{C}$ *stands for the appropriate space of continuous functions of a single variable.*

LEMMA 2. *The function* $\alpha \circ \phi + \beta \circ \psi$ *of* $\mathcal{B}_n$ *is a best approximation to* $f \in \mathcal{C}_n$ *if* α *minimizes the functional* ω,

$$\omega(f - \alpha \circ \phi \,|\, \psi) = \inf_{a \in \mathbb{C}} \omega(f - a \circ \phi \,|\, \psi),$$

and $\beta \circ \psi$ is defined for each l_ψ as

$$\beta \circ \psi = \tfrac{1}{2}[\max_{l\psi}(f - \alpha \circ \phi) + \min_{l\psi}(f - \alpha \circ \phi).$$

Since ω measures the oscillation of f on the partition L_ψ (of E_n), we agree to call α a best ω-approximation of f (see also [3]).

LEMMA 3. *Given a function* $f \in \mathcal{C}_n$, *consider the family* $\mathfrak{f}$ *whose members are defined for each admitted level set* $l_\psi(t)$ *to be the restriction of* f *to it:*

$$\mathfrak{f} = \{f_t : f_t = f \,|\, l_\psi(t)\};$$

let $\mathfrak{f}_\epsilon \subset \mathfrak{f}$ *be an ϵ-net of the family* $\mathfrak{f}$, *then*

$$\inf_{a \in \mathbb{C}} \omega(\mathfrak{f}_\epsilon - a \circ \phi \,|\, \psi) \leqq \mu(f) \leqq \inf_{a \in \mathbb{C}} \omega(\mathfrak{f}_\epsilon - a \circ \phi \,|\, \psi) + \epsilon,$$

where

$$\omega(\mathfrak{f}_\epsilon - a \circ \phi \,|\, \psi) = \max_{f_t \in \mathfrak{f}_\epsilon} \omega(f_t - a \circ \phi \,|\, l_\psi(t)).$$

LEMMA 4. *Each finite family* $\mathfrak{f}_\epsilon$ *has a best ω-approximation.*

LEMMA 5. *Corresponding to the members* $\epsilon_k > 0$ *of the convergent series* $\sum \epsilon_k$, *let* $\{\mathfrak{f}_k\}$ *be an ascending chain of ϵ_k-nets of* $\mathfrak{f}$:

$$\mathfrak{f}_1 \subset \mathfrak{f}_2 \subset \cdots \subset \mathfrak{f}_k \subset \cdots \subset \mathfrak{f}.$$

There is a uniformly convergent sequence of best ω-approximations α_k *to* $\mathfrak{f}_k$ *whose limit is a best ω-approximation to the function f attached to* $\mathfrak{f}$.

Theorem 1 was arrived at through an examination of the constructions in [2]. Specifically, the first three lemmas above correspond to the first three theorems in [2]; Lemma 5 is the counterpart of [2, Theorem 4]. The proofs of the preceding lemmas require certain modifications of the arguments in [2], based on these facts:

Given f, let $\mathfrak{f}_\epsilon = \{f_1, \cdots, f_m\}$, $f_j = f \,|\, l_\psi(t_j)$, be a fixed ϵ-net of $\mathfrak{f}$; let $\mathfrak{f}_{\epsilon k}$, $1 \leqq k \leqq m$, consist of the first k elements of $\mathfrak{f}_\epsilon$.

(i) If α_k is a best ω-approximation of $\mathfrak{f}_{\epsilon k}$, then so is $\alpha_k + c$ for each constant c.

(ii) If α_{k+1} is a best ω-approximation of f_{ek+1}, then there is a constant c such that for each point $p \in l_\psi(t_{k+1})$, $\text{sgn}\,[f_{k+1} - \alpha_{k+1} \circ \phi - c]$ $= \text{sgn}\,[\alpha_{k+1} \circ \phi + c - \alpha_k \circ \phi]$, unless $f_{k+1} = \alpha_{k+1} \circ \phi + c$ or $\alpha_k \circ \phi = \alpha_{k+1} \circ \phi$ $+ c$.

2. **On the choice of ϕ and ψ.** The following propositions are conveniently formulated with this notation:

Consider two functions, ψ_1 and ψ_2, of $\mathbb{C}_n$: if L_{ψ_1} is a refinement of L_{ψ_2}, when these are regarded as partitions of E_n, then we write $\psi_2 \succ \psi_1$ (see [1]). Designating now best approximations of f in $\mathbb{G}_n$ relative to the fixed functions ϕ and ψ by $\mu(f; \phi, \psi)$, we state

THEOREM 3. *If $\phi_2 \succ \phi_1$ and $\psi_2 \succ \psi_1$, then*

$$\mu(f; \phi_2, \psi_2) \geqq \mu(f; \phi_1, \psi_1)$$

for each $f \in \mathbb{C}_n$.

THEOREM 4. *If $\phi \succ \psi$, then*

$$\mu(f; \phi, \psi) = \tfrac{1}{2}\omega(f \mid \psi).$$

THEOREM 5. *For each $p \in E_n$ and $i = 1, 2$, let U_{ip} designate the intersection of level sets l_{ϕ_i} and l_{ψ_i} containing p (that is, we remove now the restriction imposed in the fixed functions in Theorem 1). If $U_{2p} \supset U_{1p}$ for each $p \in E_n$, then the conclusion of Theorem 3 remains valid for each $f \in \mathbb{C}_n$.*

Theorem 3 is a generalization of [1, Corollary 3.2]. To prove it, consider first the case when $\phi_1 = \phi_2$. Given f, let $\{a_k\}$ be a sequence of members of $\mathbb{C}$ such that

$$\lim_{k \to \infty} \omega(f - a_k \circ \phi_2 \mid \psi_2) = \mu(f; \phi_2, \psi_2):$$

clearly, we are guaranteed the existence of such sequences. Owing to [1, Corollary 2.1], we have the inequality

$$\omega(f - a_k \circ \phi_2 \mid \psi_2) \geqq \omega(f - a_k \circ \phi_2 \mid \psi_1)$$

for each fixed k, and consequently this remains true as we let $k \to \infty$; while we are not assured the convergence of the sequence whose members are $\omega(f - a_k \circ \phi_2 \mid \psi_1)$, we can state that

$$\mu(f; \phi_2, \psi_2) \geqq \limsup_{k \to \infty} \omega(f - a_k \circ \phi_2 \mid \psi_1).$$

If it happens that the right side of this inequality equals $\mu(f; \phi_2, \psi_1)$ then we have established this case; otherwise, we consider a sequence, $\{c_k\}$, of functions $c_k \in \mathbb{C}$, subject to the specification

$$\lim_{k \to \infty} \omega(f - c_k \circ \phi_2 \,|\, \psi_1) = \mu(f; \phi_2, \psi_1).$$

A comparison of the last two inequalities reveals at once that

$$\mu(f; \phi_2, \psi_2) \geqq \mu(f; \phi_2, \psi_1),$$

thereby disposing of the case in question.

The proof of the theorem is completed by applying Lemma 1 above to the last inequality, and using the fact that

$$\mu(f, \phi, \psi) = \inf_{a \in \mathcal{C}} \inf_{b \in \mathcal{C}} \|f - a \circ \phi - b \circ \psi\| = \inf_{b \in \mathcal{C}} \inf_{a \in \mathcal{C}} \|f - a \circ \phi - b \circ \psi\|.$$

Theorem 4 follows at once from the fact that under these circumstances $\phi = g \circ \psi$ for some function $g \in \mathcal{C}$ (see [1]).

Theorem 5 is verified with the basic inequality $\omega(f \,|\, S) \geqq \omega(f \,|\, T)$ whenever $S \supset T$ and the fact that if $l_\phi \cap l_\psi = S$, then

$$\mu(f \,|\, S; \phi, \psi) = \tfrac{1}{2}\omega(f \,|\, S).$$

In particular, this theorem implies that in order to optimize our best approximations, all members of L_ϕ should intersect those of L_ψ in at most one point.

In connection with these results we pose this problem:

It is well known that certain partitions of E_n cannot qualify for membership in the family of level sets of a continuous function. Thus, for example, when $L_f = E_n$, then f is at most continuous almost everywhere. We ask, therefore, for necessary and sufficient conditions for a partition P of E_n to admit a function $f \in \mathcal{C}_n$, such that $P = L_f$; similarly, under what circumstances can the function so associated with P be continuous merely almost everywhere?

BIBLIOGRAPHY

1. D. A. Sprecher, *On best approximations in several variables*, J. Reine Angew. Math (to appear).

2. ———, *On best approximations of functions of two variables*, (to appear).

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