AUTOMORPHISMS OF COMPACT RIEMANN SURFACES AND THE VANISHING OF THETA CONSTANTS

BY H. M. FARKAS[1]

Communicated by Maurice Heins, October 13, 1966

I. It is the purpose of this note to announce a theorem which shows that there exists a connection between automorphisms of compact Riemann surfaces and the vanishing of Riemann theta constants. In particular we shall outline the proof of the following theorem:

THEOREM 1. *Let S be a compact Riemann surface of genus $2g-1$, $g\geqq 2$, which permits a conformal fixed point free involution T. Let $\gamma_1, \cdots, \gamma_{2g-1}; \delta_1, \cdots, \delta_{2g-1}$ be a canonical dissection of S and let T be such that $T(\gamma_1)$ is homologous to γ_1, $T(\delta_1)$ is homologous to δ_1, $T(\gamma_i)$ is homologous to γ_{g+i-1} and $T(\delta_i)$ is homologous to δ_{g+i-1}, $i=2, \cdots, g$. Then, there exist at least $2^{g-2}(2^{g-1}-1)$ half integer theta characteristics $\epsilon_1, \cdots$ such that $\theta_{\epsilon_1}(0)=\theta_{\epsilon_2}(0)= \cdots =0$ and the order of the zero is $\geqq 2$.*

The proof of the theorem rests on the fact that S is a two sheeted nonbranched covering of a compact Riemann surface of genus g and the following lemma which was proved in [1].

LEMMA 1. *Let ζ,ω be equivalent special divisors of degree $G-1$ on a compact Riemann surface S of genus G. Then if $i(\zeta\omega)=1$, where "i" is the index of specialty of the divisor, there exists a half integer characteristic ϵ corresponding to the divisor ζ such that $\theta_\epsilon(0)=0$ and the order of the zero is $\geqq 2$. θ is of course the Riemann theta of S.*

II. Let $\hat{S}=S/T$ denote the compact Riemann surface of genus g which is covered by S. Then all the functions and differentials which exist and are well defined on $\hat{S}$ may be lifted to S and are well-defined objects thereon. As a matter of fact all such lifted functions and differentials will be invariant under the involution T and conversely all objects on S which are invariant under T are well defined on $\hat{S}$. There are, however, objects which are not well defined on $\hat{S}$ but are well defined on S. For example, let $\hat{\theta}_\alpha$ and $\hat{\theta}_\beta$ be two odd Riemann thetas associated with $\hat{S}$ such that $\alpha+\beta\equiv\delta$ where δ is the characteristic $(0, \cdots, 0; \frac{1}{2}, 0 \cdots 0)$, α, β half integer characteristics. Then the quotient $\hat{\theta}_\alpha/\hat{\theta}_\beta$ is not well defined on $\hat{S}$ for analytic continuation of

[1] Research partially supported by NSF GP 3452.

$\hat{\theta}_\alpha/\hat{\theta}_\beta$ around the cycle δ_1 on $\hat{S}$ carries $\hat{\theta}_\alpha/\hat{\theta}_\beta$ into $-\hat{\theta}_\alpha/\hat{\theta}_\beta$. Such functions will be called multiplicative functions and they will be discussed in detail by H. E. Rauch and the author in a forthcoming paper [2]. The crucial thing here is that $\hat{\theta}_\alpha/\hat{\theta}_\beta$ is well defined on S and is a single valued meromorphic function on S with $2g-2$ zeros and $2g-2$ poles. Utilizing now the well-known result that associated with an odd theta there is an Abelian differential of first kind with double zeros, we are through. For let $d\hat{\omega}_\beta$ be the Abelian differential of first kind associated with $\hat{\theta}_\beta$ on $\hat{S}$, and let $d\omega_\beta$ be the lifted differential on S. $(\hat{\theta}_\alpha/\hat{\theta}_\beta)d\omega_\beta$ is then an Abelian differential of first kind on S and its zeros are precisely the same as the zeros and poles of $\hat{\theta}_\alpha/\hat{\theta}_\beta$. Hence we have shown that on S there exist equivalent special divisors ζ, ω of degree $G-1=2g-2$, the respective zero and polar divisors of $\hat{\theta}_\alpha/\hat{\theta}_\beta$, such that $i(\zeta\omega)=1$. Hence by the lemma there exists a half integer characteristic ϵ such that $\theta_\epsilon(0)=0$ and the order of the zero is $\geqq 2$.

It is now easy to compute a lower bound for the number of half integer characteristics ϵ_i for which $\theta\epsilon_i(0)=0$ and the order of the zero is $\geqq 2$. There are at least as many half integer characteristics as there are pairs of odd theta characteristics α, β such that $\alpha+\beta\equiv\delta$. This number is clearly equal to $2^{g-2}(2^{g-1}-1)$. The proof of Lemma 1 implies that different pairs (α, β) correspond to different ϵ.

COROLLARY 1. *If the genus of S is 3 then S is hyperelliptic.*

PROOF. By Theorem 1, there exists one half integer characteristic ϵ such that $\theta_\epsilon(0)=0$ and the order of the zero is $\geqq 2$. However, it is known that for a compact Riemann surface of genus 3, 2 is an upper bound for the order of vanishing of a theta [3]. Hence the characteristic ϵ is an even characteristic and it is well known that a necessary and sufficient condition for a compact Riemann surface of genus 3 to be hyperelliptic is that one even theta vanish at the origin.

The result may also be obtained directly in this case by observing that the lifted function $\hat{\theta}_\alpha/\hat{\theta}_\beta$ is a function with two poles.

REFERENCES

1. H. M. Farkas, *Special divisors and analytic subloci of Teichmueller space*, Amer. J. Math. **88** (1966), 881–901.

2. H. E. Rauch and H. M. Farkas, *Multiplicative functions on compact Riemann surfaces* (in preparation).

3. J. Lewittes, *Riemann surfaces and the theta function*, Acta Math. **111** (1964), 37–61.

JOHNS HOPKINS UNIVERSITY AND
YESHIVA UNIVERSITY