ON LIE ALGEBRAS OF TYPE E_6

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Introduction. In this note, we investigate, omitting details, the structure of Lie algebras of type E_6 over arbitrary fields of characteristic other than two or three, introducing certain invariants for such algebras and studying the implications these invariants have for the structure of the algebras in question. As a consequence of this investigation, we obtain, producing constructively a representative of each isomorphism class, a complete classification of algebras of type E_6 over finite, real closed, or p-adic fields, as well as partial results for algebraic number fields. Since every Lie algebra of type E_6 over Φ has a finite, Galois, splitting field $P \supseteq \Phi$, [1], we restrict our attention, without loss of generality, to a particular pair of fields P and Φ, P finite, Galois over Φ with group G, and to the collection of Lie algebras of type E_6 over Φ which are split by P.

Realization of the split E_6. Let $\mathfrak{J}_0$ be a split exceptional central simple Jordan algebra over Φ, $\mathfrak{J} = \mathfrak{J}_0 \otimes_\Phi P$, and V the P-space of all

$$x = \begin{pmatrix} \alpha_1 & a_1 \\ a_2 & \alpha_2 \end{pmatrix}, \quad \alpha_i \in P, \ a_i \in \mathfrak{J}.$$

V, with quartic form

$$q(x) = 8(a_1 \times a_1, a_2 \times a_2) - 8\alpha_1 N(a_1) - 8\alpha_2 N(a_2) - 2((a_1, a_2) - \alpha_1\alpha_2)^2,$$

(a, b) the trace bilinear form of $\mathfrak{J}$, $N(a)$ the generic norm on $\mathfrak{J}$, $\times$ the product defined in [4], is a module for the split Lie algebra of type E_7 [4], [8]. The algebra $\mathfrak{L}(V, V_0) = \{L \in \mathrm{Hom}(V, V) \mid V_0 L = 0, L \text{ skew with respect to the linearized } q(x)\}$, V_0 the subspace of V of diagonal elements, is a split Lie algebra of type E_6. The semiautomorphisms of $\mathfrak{L}(V, V_0)$ are described by

THEOREM 1. *$A(s)$ is an s-semiautomorphism of $\mathfrak{L}(V, V_0)$ if and only if there is a permutation π of $\{1, 2\}$, an element $\gamma \in P^*$, and an s-semilinear transformation $T(s)$ on $\mathfrak{J}$ satisfying $N(xT(s)) = \mu N(x)^s$ for all $x \in \mathfrak{J}$, such that*

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$$LA(s) = [\pi, \gamma, T(s)]^{-1}L[\pi, \gamma, T(s)] \overset{\text{def}}{=} L[\pi, \gamma, T(s)]^{\nu}$$

for all $L \in \mathfrak{L}(V, V_0)$, $[\pi, \gamma, T(s)]$ *the s-semilinear transformation on V such that*

$$\begin{pmatrix} \alpha_1 & a_1 \\ a_2 & \alpha_2 \end{pmatrix} [\pi, \gamma, T(s)] = \begin{pmatrix} \overset{s}{\alpha}_{1\pi}\mu^{-1}\gamma^2 & a_{1\pi}T(s) \\ a_{2\pi}T(s)^{*-1}\gamma & \overset{s}{\alpha}_{2\pi}\mu\gamma^{-1} \end{pmatrix}$$

*where * is the transpose with respect to* (a, b) *in* $\mathfrak{J}$.

Invariants. If $\mathfrak{L}$ is of type E_6 over Φ, $\mathfrak{L}_P = \mathfrak{L} \otimes_\Phi P$ split, we may, up to isomorphism, assume that $\mathfrak{L}$ is a Φ-form of $\mathfrak{L}(V, V_0)$ and hence, by standard results, there is a homomorphism $s \to A(s)$ of G into the group of s-semiautomorphisms of $\mathfrak{L}(V, V_0)$ such that

$$\mathfrak{L} = \{L \in \mathfrak{L}(V, V_0) \mid LA(s) = L \text{ for all } s \in G\}.$$

By Theorem 1, $A(s) = [\pi(s), \gamma(s), T(s)]^{\nu}$ for each s and one sees easily that $s \to \pi(s)$ is a homomorphism of G into S_2 which is invariant under isomorphism of $\mathfrak{L}$. We call $\mathfrak{L}$ of type $E_{6\mathrm{I}}$ if $\pi(G) = 1$, of type $E_{6\mathrm{II}}$ if $\pi(G) = S_2$. If $\mathfrak{L}$ is of type $E_{6\mathrm{II}}$, there is a unique quadratic extension Δ of Φ (the canonical $E_{6\mathrm{I}}$ extension of Φ for $\mathfrak{L}$) such that $\mathfrak{L}_\Delta$ is of type $E_{6\mathrm{I}}$.

Considering $\mathfrak{L}$ as Φ-subalgebra of $\mathfrak{L}(V, V_0) \subseteq \mathrm{Hom}(V, V)$, we define $\mathfrak{L}^*$ to be the enveloping associative algebra of $\mathfrak{L}$ in $\mathrm{Hom}(V, V)$ and observe that $\mathfrak{L}^*$ is an invariant of the isomorphism class of $\mathfrak{L}$.

Finally, defining $\mathfrak{M}'(V, V_0) = \{[1, \gamma, T] \mid T \text{ linear}, \gamma \in P^*\}$ and $K = \{[1, \gamma, \alpha I] \mid I \text{ the identity on } \mathfrak{J}, \alpha \in P^*\}$ we have an exact sequence

$$(1) \qquad 1 \to K \to \mathfrak{M}'(V, V_0) \overset{\nu}{\to} \mathrm{Aut}'\, \mathfrak{L}(V, V_0) \to 1$$

where $\mathrm{Aut}'\, \mathfrak{L}(V, V_0)$ is the image of $\mathfrak{M}'(V, V_0)$ under ν. This can be made into a sequence of G-groups by defining the action of $s \in G$ on $\mathfrak{M}'(V, V_0)$ (resp. $\mathrm{Aut}'\, \mathfrak{L}(V, V_0)$) to be conjugation by $[\pi(s), 1, I(s)]$ (resp. $[\pi(s), 1, I(s)]^{\nu}$), $I(s)$ the s-semilinear extension of the identity on $\mathfrak{J}_0$ to $\mathfrak{J}$, $s \to \pi(s)$ the homomorphism associated with $\mathfrak{L}$. Since K is contained in the center of $\mathfrak{M}'(V, V_0)$ and one can show $H^1_\pi(G, K) = 1$, we have the exact cohomology sequence

$$(2) \quad 1 \to H^1_\pi(G, \mathfrak{M}'(V, V_0)) \to H^1_\pi(G, \mathrm{Aut}'\, \mathfrak{L}(V, V_0)) \to H^2_\pi(G, K).$$

Identifying, in the usual manner [6], the elements of $H^1_\pi(G, \mathrm{Aut}'\, \mathfrak{L}(V, V_0))$ with the equivalence classes of isomorphic forms of $\mathfrak{L}(V, V_0)$ having associated homomorphism $\pi: G \to S_2$, we define $\Gamma(\mathfrak{L}) \in H^2_\pi(G, K)$ to be the image of the element of $H^1_\pi(G, \mathrm{Aut}'\, \mathfrak{L}(V, V_0))$

corresponding to the class containing $\mathfrak{L}$, in (2). $\Gamma(\mathfrak{L})$is not, in general, an invariant of the complete isomorphism class containing $\mathfrak{L}$.

We have the following results describing and relating the invariants.

THEOREM 2. (a) *$\mathfrak{L}$ is of type E_{6I} over Φ if and only if $\mathfrak{L}^* \cong \mathfrak{B} \oplus \mathfrak{B}'$, $\mathfrak{B}'$ antiisomorphic to $\mathfrak{B}$, $\mathfrak{B}$ central simple of exponent three, degree 27 over Φ.*

(b) *$\mathfrak{L}$ is of type E_{6II} with canonical E_{6I} extension Δ of Φ if and only if $\mathfrak{L}^* \cong \mathfrak{B}$, $\mathfrak{B}$ central simple associative of exponent three, degree 27 over Δ with involution of the second kind over Φ.*

THEOREM 3. *$\Gamma(\mathfrak{L})=1$ if and only if $\mathfrak{L}^*$ is isomorphic to either $\Phi_{27} \oplus \Phi_{27}$ or to Δ_{27}, depending on the E_6 type of $\mathfrak{L}$.*

Since $\mathfrak{L}^*$ is invariant under isomorphism, so is the property $\Gamma(\mathfrak{L})=1$.

Construction of algebras. Let $\mathfrak{J}$ be an arbitrary exceptional central simple Jordan algebra over Φ, $N(x)$ the generic norm form of $\mathfrak{J}$, $N(x, y, z)$ the linearized norm form. The set of all $L \in \mathrm{Hom}(\mathfrak{J}, \mathfrak{J})$ such that $N(xL, x, x)=0$ for all $x \in \mathfrak{J}$ is an algebra $\mathfrak{L}(\mathfrak{J})$ of type E_6 [3] and can be written as the direct sum $T(\mathfrak{J})+\theta(\mathfrak{J})$, $T(\mathfrak{J})$ the set of right multiplications by elements of trace zero in $\mathfrak{J}$, $\theta(\mathfrak{J})$ the derivation algebra of $\mathfrak{J}$. Since $[T(\mathfrak{J}), T(\mathfrak{J})] \subseteq \theta(\mathfrak{J})$, $[T(\mathfrak{J}), \theta(\mathfrak{J})] \subseteq T(\mathfrak{J})$, Albert has observed that, for $\lambda \in \Phi$, $\lambda^{1/2} \notin \Phi$, $\mathfrak{L}(\mathfrak{J})_\lambda = \lambda^{1/2} T(\mathfrak{J})+\theta(\mathfrak{J})$ with the natural multiplication is again an algebra of type E_6 over Φ. Finally, adapting a construction of Tits [9], we obtain an algebra $\mathfrak{L}(\mathfrak{C}, \mathfrak{A})$, $\mathfrak{C}$ a Cayley algebra over Φ, $\mathfrak{A}$ a central simple associative algebra of degree three over Φ with generic trace form $T(x)$, by defining on the vector space $\theta(\mathfrak{C})+\mathfrak{C} \otimes \mathfrak{A}_0$, $\mathfrak{A}_0$ the kernel of $T(x)$, a multiplication $\langle x, y\rangle$ such that

$$
\begin{aligned}
\langle D_1, D_2\rangle &= [D_1, D_2],\\
\langle a_1 \otimes x_1, D_1\rangle &= a_1 D_1 \otimes x_1,\\
\langle a_1 \otimes x_1, a_2 \otimes x_2\rangle &= [a_1, a_2] \otimes (x_1 \cdot x_2 - (x_1, x_2)1)\\
&\quad + a_1 \cdot a_2 \otimes [x_1, x_2] + 1/3(x_1, x_2) D_{a_1, a_2}
\end{aligned}
$$

for $a_i \in \mathfrak{C}$, $x_i \in \mathfrak{A}$, $D_{a_1,a_2}=[a_1, a_2]_r-[a_1, a_2]_l-3[(a_1)_l, (a_2)_r]$ in $\mathfrak{C}$, a_r and a_l denoting right and left multiplication in $\mathfrak{C}$, $[u, v]=uv-vu$, $u \cdot v = \frac{1}{2}(uv+vu)$, $(x_1, x_2)=T(x_1 x_2)$, 1 the identity of $\mathfrak{A}$, $D_i \in \theta(\mathfrak{C})$ the derivation algebra of $\mathfrak{C}$.

Identifying these algebras with suitable Φ-subalgebras of $\mathfrak{L}(V, V_0)$ we have

THEOREM 4. (a) $\mathfrak{L}(\mathfrak{J})$ *is a Lie algebra of type* $E_{6\mathrm{I}}$ *with* $\Gamma(\mathfrak{L}(\mathfrak{J}))=1$.

(b) $\mathfrak{L}(\mathfrak{J})_\lambda$ *is a Lie algebra of type* $E_{6\mathrm{II}}$ *with* $\Gamma(\mathfrak{L}(\mathfrak{J})_\lambda)=1$.

(c) $\mathfrak{L}(\mathfrak{C}, \mathfrak{A})$ *is a Lie algebra of type* $E_{6\mathrm{I}}$ *with* $\mathfrak{L}^*\cong\mathfrak{B}+\mathfrak{B}'$, $\mathfrak{B}$ *equivalent to* $\mathfrak{A}$ *in the Brauer group.*

Theorems 2, 3, and 4 imply that, if $\mathfrak{A}$ is a division algebra over Φ, $\mathfrak{J}$, $\mathfrak{J}'$ exceptional central simple Jordan algebras over Φ, then

COROLLARY. $\mathfrak{L}(\mathfrak{J})$, $\mathfrak{L}(\mathfrak{J}')_\lambda$, $\mathfrak{L}(\mathfrak{C}, \mathfrak{A})$ *are nonisomorphic Lie algebras of type* E_6.

We note that the converse of Theorem 4, (a) is true in general, but that we have been unable to establish converses for Theorem 4, (b) and (c) in general.

Special fields. We make the additional assumption in this section that there are no exceptional Jordan division algebras over Φ and obtain a converse to Theorem 4, yielding

THEOREM 4′. *Let* $\mathfrak{L}$ *be a Lie algebra of type* E_6 *over* Φ. *Then*

(a) $\mathfrak{L}\cong\mathfrak{L}(\mathfrak{J})$ *if and only if* $\Gamma(\mathfrak{L})=1$, $\mathfrak{L}$ *of type* $E_{6\mathrm{I}}$.

(b) $\mathfrak{L}\cong\mathfrak{L}(\mathfrak{C}, \mathfrak{A})$ *if and only if* $\mathfrak{L}^*=\mathfrak{B}+\mathfrak{B}'$, $\mathfrak{B}$ *of index* $\leqq 3$.

(c) *If every algebra of type* $E_{6\mathrm{II}}$ *over* Φ *is split by its canonical* $E_{6\mathrm{I}}$ *extension, then* $\mathfrak{L}\cong\mathfrak{L}(\mathfrak{J})_\lambda$ *if and only if* $\Gamma(\mathfrak{L})=1$, $\mathfrak{L}$ *of type* $E_{6\mathrm{II}}$.

Since for Φ finite, real closed, or p-adic, the condition of Theorem 4′, (c), as well as the general condition of this section, is satisfied, every algebra of type E_6 over such a field is isomorphic to some $\mathfrak{L}(\mathfrak{J})$, $\mathfrak{L}(\mathfrak{J})_\lambda$ or $\mathfrak{L}(\mathfrak{C}, \mathfrak{A})$ and, in fact, the latter case can occur only in case Φ is p-adic. One can easily enumerate the nonisomorphic algebras of these kinds, obtaining results agreeing with those in [5], [2], and [7]. If Φ is an algebraic number field, the condition of Theorem 4′, (c), does not hold and we can show only that the algebras of type $E_{6\mathrm{I}}$ over Φ must be isomorphic to some $\mathfrak{L}(\mathfrak{J})$ or $\mathfrak{L}(\mathfrak{C}, \mathfrak{A})$.

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