A FOURIER SERIES METHOD FOR MEROMORPHIC AND ENTIRE FUNCTIONS

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We associate with a function f, meromorphic in the finite complex plane, the Fourier expansion

$$\log |f(re^{i\theta})| = \sum c_k(r)e^{ik\theta}.$$

The growth of f, measured by the Nevanlinna characteristic $T(r, f)$, and the growth of the coefficients $c_k(r)$ are closely related, even in the case of functions of infinite order. The $c_k(r)$ can be expressed in terms of the local behaviour of f near the origin, as well as the distribution of the zeros and poles of f. Thus, results on the zero and pole distribution of meromorphic functions of a given rate of growth are obtained that involve the angular distribution as well as the radial distribution. In particular, entire functions with prescribed zeros are constructed without the use of canonical products. Also, it is proved that each meromorphic function of a reasonably general rate of growth is the quotient of two entire functions of the same rate of growth. Some of the results in this note were obtained for a special case in [2].

DEFINITION. *A growth function $\lambda(r)$ is a function defined for $0 \leqq r < \infty$ that is positive, nondecreasing, and continuous.*

DEFINITION. *A meromorphic function f is said to be of finite λ-type if there exist constants A and B such that $T(r, f) \leqq A\lambda(Br)$.*

REMARK. *An entire function f is of finite λ-type if and only if there exist constants A and B such that $|f(z)| \leqq \exp(A\lambda(B|z|))$.*

In case $\lambda(r) = r^\rho$, $\rho > 0$, then the functions of finite λ-type are precisely the functions of growth at most order ρ, finite type. Our considerations include functions $f(z)$ that grow like $\exp(\exp(\exp(|z|)))$ for example.

We consider sequences $Z = \{z_n\}$ of complex numbers, z_n distinct from 0, such that $|z_n| \to \infty$ as $n \to \infty$.

DEFINITION. *We say that Z is of finite λ-density if there exist constants A and B such that $N(r, Z) \leqq A\lambda(Br)$, where*

$$N(r, Z) = \sum_{|z_n| \leqq r} \log\left(\frac{r}{|z_n|}\right).$$

DEFINITION. *We say that Z is λ-balanced if there exist constants A and B such that for $k = 1, 2, 3, \cdots$*

$$\left|\frac{1}{k}\sum_{r_1<|z_n|\leqq r_2}\left(\frac{1}{z_n}\right)^k\right| \leqq \frac{A\lambda(Br_1)}{(r_1)^k}+\frac{A\lambda(Br_2)}{(r_2)^k},$$

whenever $0\leqq r_1\leqq r_2$.

THEOREM. *A sequence Z is the precise sequence of zeros of some entire function f of finite λ-type if and only if Z is of finite λ-density and is λ-balanced.*

This result generalizes a well-known theorem of Lindelöf [**1**, p. 27] that considers the case $\lambda(r)=r^\rho$, since it is easy to see that our conditions reduce to the conditions of Lindelöf in this case. Namely, a sequence Z of finite r^ρ density is already r^ρ-balanced if ρ is not an integer, and if ρ is an integer, it is r^ρ balanced if and only if $\sum(z_n)^{-\rho}$ is a bounded function of r, where the sum is taken over all those z_n for which $|z_n|\leqq r$.

THEOREM. *A sequence Z is the precise sequence of zeros of some meromorphic function f of finite λ-type if and only if Z is of finite λ-density.*

DEFINITION. *The growth function λ is regular if each sequence Z of finite λ-density has a supersequence $Z'\supseteq Z$ of finite λ-density that is also λ-balanced.*

THEOREM. *Each meromorphic function f of finite λ-type is the quotient of two entire functions of finite λ-type if and only if λ is regular.*

PROPOSITION. *If λ is slowly increasing in the sense that* $\lambda(2r)/\lambda(r)$ *is bounded, or if* $\log\lambda(e^x)$ *is a convex function of x, then λ is regular.*

The fundamental theorem of this paper, on which all the above results are based, is the following.

THEOREM. *Suppose that f is a meromorphic function,* $f(0)\neq 0,\ \infty$, *whose zero set has finite λ-density. If f is of finite λ-type, then*

$$(1)\qquad |c_k(r)|\leqq\frac{A\lambda(Br)}{|k|+1},\qquad k=0,\pm 1,\pm 2,\cdots.$$

On the other hand, if

$$(2)\qquad |c_k(r)|\leqq A\lambda(Br),\qquad k=0,\pm 1,\pm 2,\cdots,$$

then f is of finite λ-type.

REMARK. *Since the necessary Condition* (1) *is stronger than the sufficient Condition* (2), *it follows that both* (1) *and* (2) *are separately necessary and sufficient.*

THEOREM. *Let f be an entire function. In order that f be of finite* λ-*type,* (1) *is a necessary condition, and* (2) *is a sufficient condition, so that both* (1) *and* (2) *are each separately necessary and sufficient.*

On applying results from the theory of Fourier series, we obtain the following as corollaries of the fundamental theorem.

THEOREM. *Let f be a meromorphic function of finite* λ-*type with* $f(0)\neq 0, \infty$. *Then for each* $q\geqq 1$, $q<\infty$, *there exist constants A and B such that for all* $r>0$,

$$\left\{\frac{1}{2\pi}\int_{-\pi}^{\pi}\left|\log\left|f(re^{i\theta})\right|\right|^{q}d\theta\right\}^{1/q}\leqq A\lambda(Br).$$

THEOREM. *Let f be a meromorphic function of finite* λ-*type,* $f(0)\neq 0$, ∞. *Then for each* $\epsilon>0$, *there exist constants,* α, $\beta>0$ *such that for all* $r>0$,

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\exp\left\{\frac{\alpha\left|\log\left|f(re^{i\theta})\right|\right|}{\lambda(\beta r)}\right\}d\theta<1+\epsilon.$$

REFERENCES

1. R. P. Boas, Jr., *Entire functions*, Academic Press, New York, 1954.
2. L. A. Rubel, *A Fourier series method for entire functions*, Duke Math. J. **30** (1963), 437–442.

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