RESEARCH ANNOUNCEMENTS

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AN INEQUALITY CONCERNING MEASURES

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If μ is a complex measure (countably additive on a σ-field of subsets of some space), it is obvious that there is a measurable set E such that

$$|\mu(E)| \geqq \tfrac{1}{4}||\mu||$$

where $||\mu||$ denotes the total variation of μ. In fact a set E can be found for which

$$|\mu(E)| \geqq \frac{1}{\pi}||\mu||.$$

We shall give a simple proof of this. If μ is a vector valued measure with values in R^n (with the usual Euclidean norm) we shall show by a suitable modification of our argument that there is a set E with

$$||\mu(E)|| \geqq \frac{1}{2\pi^{1/2}} \frac{\Gamma(n/2)}{\Gamma((n+1)/2)}||\mu||.$$

Asymptotically this is $||\mu||/(2\pi n)^{1/2}$, which is much better than the obvious $||\mu||/2n$.

THEOREM 1. *Let μ be a complex valued measure of total variation* 1. *Then there is a measurable set E such that* $|\mu(E)| \geqq 1/\pi$.

PROOF. Consider first the special case where μ is a Borel measure on the unit circle of the complex plane (which we identify with the real line (mod 2π)), and is such that for every measurable set E,

$$\mu(E) = \int_E e^{i\theta}|\mu|(d\theta)$$

where $|\mu|(E)$ denotes the total variation of μ on the set E. Then

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\begin{aligned}
\max_{E \text{ measurable}} |\mu(E)| &= \max_{E \text{ measurable}} \left| \int_E e^{i\theta} |\mu| (d\theta) \right| \\
&\geqq \max_{\lambda} \left| \int_{\lambda-\pi/2}^{\lambda+\pi/2} e^{i\theta} |\mu| (d\theta) \right| = \max_{\lambda} \left| \int_{\lambda-\pi/2}^{\lambda+\pi/2} e^{i(\theta-\lambda)} |\mu| (d\theta) \right| \\
&\geqq \max_{\lambda} \int_{\lambda-\pi/2}^{\lambda+\pi/2} \operatorname{Re}(e^{i(\theta-\lambda)}) |\mu| (d\theta) \\
&\geqq \frac{1}{2\pi} \int_0^{2\pi} \int_{\lambda-\pi/2}^{\lambda+\pi/2} \operatorname{Re}(e^{i(\theta-\lambda)}) |\mu| (d\theta)\, d\lambda \\
&= \frac{1}{2\pi} \int_0^{2\pi} \int_{\theta+\pi/2}^{\theta+3\pi/2} \operatorname{Re}(e^{i(\theta-\lambda)})\, d\lambda\, |\mu| (d\theta) = \frac{1}{\pi} \cdot
\end{aligned}
$$

For the general case define f to be the Radon-Nikodym derivative $f = d\mu/d|\mu|$, and define $\nu(E) = \mu(f^{-1}(E))$ for E a Borel subset of the unit circle. The proof is easily completed by application of the special case to the measure ν.

The constant $1/\pi$ is best possible; for some measures μ there is no set E with $|\mu(E)| > 1/\pi$. We shall now determine these measures.

THEOREM 2. *Let μ be a complex valued measure with $\|\mu\| = 1$, and f the Radon-Nikodym derivative $d\mu/d|\mu|$. Then a necessary and sufficient condition that there be no measurable set E with $\mu(E) > 1/\pi$ is that*

$$\int f(t)^n |\mu| (dt) = 0$$

for $n = \pm 1, \pm 2, \pm 4, \pm 6, \pm 8, \pm \cdots$.

PROOF. Define $F_\lambda = \{t; \lambda - \pi/2 \leqq \arg f(t) \leqq \lambda + (\pi/2) (\text{mod } 2\pi)\}$.

If E is any measurable set, $\mu(E) = re^{i\lambda}$ for some choice of real numbers $r > 0$ and λ; it is then easily checked that $\operatorname{Re}(e^{-i\lambda}\mu(F_\lambda)) \geqq r$. Thus $|\mu(E)| \leqq 1/\pi$ for all measurable sets E if and only if $\operatorname{Re}(e^{-i\lambda}\mu(F_\lambda)) \leqq 1/\pi$ for all real λ. As in the proof of Theorem 1, we observe that f induces a measure ν on the unit circle such that $\nu(S) = \mu(f^{-1}(S))$ for each measurable set S of the unit circle. Then

$$\operatorname{Re}(e^{-i\lambda}\mu(F_\lambda)) = \int_{\lambda-\pi/2}^{\lambda+\pi/2} \operatorname{Re}(e^{i(\theta-\lambda)}) |\nu| (d\theta).$$

But this is a continuous function of λ whose mean for $0 \leqq \lambda \leqq 2\pi$ was shown in the proof of Theorem 1 to be $1/\pi$. Thus it never exceeds $1/\pi$ in value if and only if it is constant and a continuous function on the interval $[0, 2\pi]$ is constant if and only if its nonzero

Fourier coefficients vanish. Moreover we may interpret the function $\operatorname{Re}(e^{-i\lambda}\mu(F_\lambda))$ as the convolution of the measure $|\nu|$ with the function defined to be $\operatorname{Re}(e^{i\lambda})$ for $-\pi/2 \leqq \lambda \leqq \pi/2$, and zero elsewhere on the interval $[-\pi, \pi]$, and then extended to a periodic function. With this interpretation we see that $\operatorname{Re}(e^{-i\lambda}\mu(F_\lambda))$ has vanishing nonzero Fourier coefficients if and only if the nth Fourier-Stieltjes coefficient of the measure $|\nu|$ vanishes for $n = \pm 1, \pm 2, \pm 4, \pm 6, \cdots$. But the nth Fourier-Stieltjes coefficient of $|\nu|$ is

$$\int_0^{2\pi} e^{in\theta} |\nu| (d\theta) = \int f(t)^n |\mu| (dt).$$

The proof is thus complete. A final remark: the vanishing of the n-th Fourier-Stieltjes coefficients of $|\nu|$ for n even, $n \neq 0$, means

$$\tfrac{1}{2}(|\nu| (d\theta) + |\nu| (d(\pi + \theta))) = d\theta/2\pi,$$

and thus implies that $|\nu|$ is absolutely continuous with respect to Lebesgue measure.

Professor S. Kakutani has suggested the following geometric proof of Theorem 1. The condition that $\|\mu\| = 1$ is equivalent to the condition that the convex hull of the range of μ have perimeter 2, a fact which is easily seen for a finite measure space and easily deduced from this for a general measure space. (If μ is completely nonatomic its range is already a convex set, by a theorem of Liapunoff, see [2]). We thus consider the following isoperimetric problem; "Of all convex sets of perimeter 2, which one is contained in the smallest disk with centre 0?" It is easily seen that the answer is the disk of radius $1/\pi$, and from this fact Theorem 1 follows.

If μ is merely a finitely additive set function (complex valued of total variation 1) it is easily deduced from Theorem 1 (for finite measure spaces) that for any $\epsilon > 0$ there is a measurable set E with

$$\mu(E) \geqq 1/\pi - \epsilon.$$

It may be asked how the constant $1/\pi$ must be changed if instead of the usual Euclidean distance, the plane is given a different norm $\|\cdot\|$. Using the approach of Professor Kakutani it is not difficult to show that the constant becomes $2/s$, where s is the perimeter of the unit ball $\{x; \|x\| \leqq 1\}$, s being measured with the distance function obtained from the norm $\|\cdot\|$. This perimeter is smallest when the unit ball is a regular hexagon: in this case the perimeter is 6.

We now consider the vector valued case.

THEOREM 3. *Let ν be a measure with values in R^n and such that $\|\nu\|=1$. Then there is a measurable set E with*

$$\|\nu(E)\| \geqq \left(\Gamma\left(\frac{n}{2}\right)\right) \Big/ \left(2\pi^{1/2}\Gamma\left(\frac{n+1}{2}\right)\right).$$

PROOF. We introduce the following notation. Denote by S the unit sphere in R^n, and by S^+ the set $\{x;\ x\in S,\ x_1\geqq 0\}$ (x_1 being the first co-ordinate of x). Denote by m the usual spherical mean on S; that is the uniformly distributed measure on S with $m(S)=1$. Let G denote the orthogonal group acting in R^n. Let x_0 be the point $(1, 0, 0, \cdots, 0)$ of S and let K be the group of those elements of G which fix x_0. We shall use the notation m_K for Haar measure on K, and m_G for the Haar measure on G (with the usual normalization for compact groups). For each positive measure μ on S define a positive measure $\tilde{\mu}$ on G as follows: if f is a continuous function on G define $\tilde{f}$ on S by

$$\tilde{f}(gx_0) = \int_K f(gk)m_K(dk)$$

and define $\tilde{\mu}$ to be that measure on G such that for any continuous function f on G

$$\int_G f(g)\tilde{\mu}(dg) = \int_S \tilde{f}(x)\mu(dx).$$

It is obvious that $\tilde{m}=m_G$, and that for any continuous function h on S,

$$\int_S h(x)\mu(dx) = \int_G h(gx_0)\tilde{\mu}(dg).$$

Finally, denote by ϕ the continuous function on S defined by $\phi(x)=\max(x_1, 0)$. As in the proof of Theorem 1 there is no loss of generality in assuming that the measure ν is a Borel measure on S such that

$$\nu(E) = \int_E x\mu(dx)$$

for each measurable set E, where μ is a probability measure on S. But then

$$
\begin{aligned}
\max_{E \text{ measurable}} \left\| \int_E x\mu(dx) \right\| &\geqq \max_{g\in G} \left\| \int_{g^{-1}S+} x\mu(dx) \right\| \\
&= \max_{g\in G} \left\| g \int_{g^{-1}S+} x\mu(dx) \right\| = \max_{g\in G} \left\| \int_{g^{-1}S+} gx\mu(dx) \right\| \\
&\geqq \max_{g\in G} \int_{g^{-1}S+} (gx)_1\mu(dx) = \max_{g\in G} \int_{g^{-1}S+} \phi(gx)\mu(dx) \\
&= \max_{g\in G} \int_S \phi(gx)\mu(dx) = \max_{g\in G} \int_G \phi(gg'x_0)\tilde{\mu}(dg') \\
&\geqq \int_G \int_G \phi(gg'x_0)\tilde{\mu}(dg')m_G(dg) = \int_G \int_G \phi(gg'x_0)m_G(dg)\tilde{\mu}(dg') \\
&= \int_G \int_G \phi(gx_0)m_G(dg)\tilde{\mu}(dg') = \int_G \phi(gx_0)m_G(dg) \\
&= \int_S \phi(x)m(dx) = \int_{S+} x_1 m(dx) = \frac{1}{2\pi^{1/2}} \frac{\Gamma(n/2)}{\Gamma((n+1)/2)}.
\end{aligned}
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As in Theorem 1 this is best possible, as the case $\mu = m$ demonstrates. After the obvious modification the discussion after Theorem 2 on the case of finitely additive set functions is applicable once again.

There seems to be no satisfactory geometric proof of Theorem 3 analogous to the one suggested by Professor Kakutani for Theorem 1. However the condition $\|\nu\| = 1$ can be stated geometrically in terms of the convex hull of the range of ν. It is known that if K is any compact convex set, and if B denotes the unit ball of R^n, then the volume of $K + rB$ is a polynomial in r of degree n (see [1]). If K is the convex hull of the range of the vector valued measure ν, then the condition that $\|\nu\| = 1$ is equivalent to the coefficient of r^{n-1} in the polynomial $\mathrm{vol}(K + rB)$ being equal to the $n-1$ dimensional volume of the unit ball in R^{n-1}.

References

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2. P. R. Halmos, *The range of a vector measure*, Bull. Amer. Math. Soc. 54 (1948), 416–421.

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