ON THE SPECTRUM AND RESOLVENT OF HOMOGENEOUS ELLIPTIC DIFFERENTIAL OPERATORS WITH CONSTANT COEFFICIENTS[1]

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As is well known, in recent years tremendous progress has been made in the study of linear partial differential equations in general and elliptic equations in particular. Powerful existence and regularity theorems have been proved by a number of authors for example Browder [3], [4], Agmon, Douglis, and Nirenberg [1], and Schechter [5], [6], [7], [8]. For general boundary value problems the existence theory has been in the form of alternative theorems and has been limited to relatively compact regions in Euclidean space. To the present author's knowledge there are no general results even for constant coefficient operators in half spaces. It is the purpose of the present paper to make a very small beginning in remedying this lack.

To be precise in the half space $\Omega=\{x\in \boldsymbol{R}^n: x_n>0\}$ let us consider the homogeneous elliptic differential operator with constant coefficients $A=\sum_{|\alpha|=2m} a_\alpha D^\alpha$ and a family of m homogeneous, constant coefficient "boundary operators" B_j, $0\leqq j\leqq m-1$. If $\lambda\in \boldsymbol{C}$, the complex numbers, we ask for necessary and sufficient conditions on λ and the B_j's in order that the map $u\rightarrow(A-\lambda)u, B_0u, \cdots, B_{m-1}u$ be a topological isomorphism of $H^{2m}(\Omega)$ onto $H^0(\Omega)\times\prod_{j=0}^{m-1} H^{2m-m_j-1/2}\Gamma$ where $\Gamma=\partial\Omega$ and $m_j<2m$ is the order of B_j. Suppose that the $\{B_j\text{'s}\}$ are such that the map $u\rightarrow(Au, B_0u, \cdots, B_{m-1}u)$ has closed graph as a map of $H^0(\Omega)\rightarrow H^0(\Omega)\times\prod_{j=0}^{m-1} H^{2m-m_j-1/2}\Gamma$ i.e. such that the usual a priori estimates are satisfied. Then we find that the necessary and sufficient condition in order that the map $u\rightarrow(A-\lambda)u, B_0u, \cdots, B_{m-1}u$ be an isomorphism is independent of the particular choice of the $\{B_j\text{'s}\}$ so long as the above mentioned operator has closed graph. As a by-product we find that under these conditions if the operator in H^{2m} defined by A and the null boundary conditions has closed range it is an isomorphism of its domain onto $H^0(\Omega)$.

We will use the following notation: $\boldsymbol{R}$ will denote the real numbers, $\boldsymbol{C}$ the complex numbers. Vectors in $\boldsymbol{R}^n$ will be denoted by $x=(x_1, \cdots, x_n)$ or by $\xi=(\xi_1, \cdots, \xi_n)$ if we refer to the usual dual space. The duality will merely be denoted by $x\xi=\sum_{j=1}^n x_j\xi_j$. If α is an n-tuple of nonnegative integers then $|\alpha|=\sum_{j=1}^n \alpha_j, \xi^\alpha=\xi_1^{\alpha 1}\cdots\xi_n^{\alpha n}$ and $D^\alpha=D_1^{\alpha_1}\cdots D_n^{\alpha_n}$ where $D_j=(1/i)(\partial/\partial x_j)$. For $u\in\mathcal{S}$ (the space

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of rapidly decreasing functions) we let $\hat{u}$ be the Fourier transform $\hat{u}(\xi)=(2\pi)^{-n/2}\int e^{-ix\xi}u(x)dx$. We use the same symbol for the Fourier-Plancherel transform. $\mathfrak{D}(\Omega)$ will denote as usual the space of infinitely differentiable functions with compact support in Ω and $\mathfrak{D}(\bar{\Omega})$ the restrictions to Ω of functions in $\mathfrak{D}(R^n)$. The completion of $\mathfrak{D}(\Omega)(\mathfrak{D}(\bar{\Omega}))$ in the norm

$$\|u\|_m = \left(\int \sum_{|\alpha|\leq m} |D^\alpha u|^2 dx\right)^{1/2}$$

will be denoted by $H_0^m(\Omega)(H^m(\Omega))$. An equivalent norm in $\boldsymbol{R}^n$ is $\|u\|_m^2=\int(1+|\xi|^2)^m|\hat{u}(\xi)|^2d\xi$. It has the added advantage of being easily generalized to m real. If $\Omega=\boldsymbol{R}^n$ it is suppressed. Constants will always be denoted by c whether or not they are the same.

Let $A=\sum_{|\alpha|=2m} a_\alpha D^\alpha$ the a_α's being possibly complex constants and let $A(\xi)=\sum_{|\alpha|=2m}a_\alpha\xi^\alpha$.

1. Definition: A is elliptic iff[2] the only $\xi\in\boldsymbol{R}^n$ for which $A(\xi)=0$ is $\xi=0$.

It follows that $|A(\xi)|\geq c|\xi|^{2m}$ where $c=\max_{|\xi|=1}|A(\xi)|$ and from this it follows that

2. Lemma. *If $A(\xi)-\lambda\neq 0$ for any $\xi\in R^n$, λ some complex number, then there is a $c>0$ such that $|A(\xi)-\lambda|\geq c(1+|\xi|^2)^m$.*

Let $\xi=(\xi',\tau)$ where $\xi'=(\xi_1,\cdots,\xi_{n-1})$ and for ξ' fixed consider the polynomial in $\tau\sharp A(\xi',\tau)-\lambda$. Suppose $\lambda\in\boldsymbol{C}$ and $A(\xi)-\lambda\neq 0$ for $\xi\in\boldsymbol{R}^n$. Let $\tau_j=\tau_j(\xi')$ be the $2m$ roots of $A(\xi',\tau)-\lambda=0$, m of the roots will have positive imaginary part. Let them be $\tau_1,\cdots,\tau_m$.

3. Lemma. (i) *There exists a constant $c>0$:*

$$|\tau_j(\xi')| \leq c(1+|\xi'|^2)^{1/2}, \qquad j=1,\cdots,2m.$$

(ii) *There exists a constant $c>0$:*

$$|\tau_j(\xi')| \geq c(1+|\xi'|^2)^{1/2}, \qquad j=1,\cdots,2m.$$

(iii) *There exits a constant $c>0$:*

$$|\operatorname{Im}\tau_j(\xi)| \geq c(1+|\xi'|^2)^{1/2}, \qquad j=1,\cdots,2m.$$

4. Proof. (i) follows from a result of Walsh [9], (ii) follows from (i) and Lemma 2, and (iii) follows from (i), (ii) and Lemma 2.

Now let B_j, $0\leqq j\leqq m-1$, be m homogeneous differential operators with constant coefficients and orders $m_j<2m$. Let $\Omega=\{x\in\boldsymbol{R}^n: x_n>0\}$ and $\Gamma=\{x\in\boldsymbol{R}^n: x_n=0\}$.

[2] iff is used to mean if and only if.

5. LEMMA. *B_j is a continuous linear map of $H^{2m}(\Omega)$ into $H^{2m-m_j-1/2}(\Gamma)$.*

6. THEOREM. *A necessary and sufficient condition that the map $u \to \{(A-\lambda)u, B_0u, \cdots, B_{m-1}u\}$ be a topological isomorphism of $H^{2m}(\Omega)$ onto $H^0(\Omega)x\prod_{j=0}^{m-1} H^{2m-m_j-1/2}\Gamma$ is that $A(\xi)-\lambda\neq 0$ for $\xi\in \boldsymbol{R}^n$ and the polynomials $\{B_j\}$ are linearly independent modulo A_λ^+.*[3] *If the condition is satisfied then there exists a $c>0$ such that for all $u\in H^{2m}(\Omega)$*

$$\|u\|_{2m} \leqq c\left[\|(A-\lambda)u\|_0 + \sum_{j=0}^{m-1}\|B_ju\|_{2m-m_j-1/2}\right] \tag{6.1}$$

PROOF. The sufficiency follows by taking Fourier transforms in the tangential variables and studying the system of ordinary differential equations thus obtained. The necessity follows by a counterexample almost identical to the one used in Agmon, Douglis, and Nirenberg [1] and the following

7. LEMMA. The map $u\to(A-\lambda)u$ of $H_0^{2m}(\Omega)$ into $H^0(\Omega)$ has a continuous inverse if and only if $A(\xi)-\lambda\neq 0$ for $\xi\in \boldsymbol{R}_n$.

If $A(\xi)-\lambda\neq 0$ for $\xi\in\boldsymbol{R}_n$ but the $\{B_j\}$ are not linearly independent it follows from the proof that not only is (6.1) not satisfied but not even the weaker in equality.

$$\|u\|_{2m} \leqq c\left[\|Au\|_0 + \|u\|_0 + \sum_{j=0}^{m-1}\|B_ju\|_{2m-m_j-1/2}\right] \text{ is satisfied.}$$

Now it is well known [1] that a necessary and sufficient condition that the latter inequality be satisfied is that A is properly elliptic and the B_j's are linearly independent modulo A^+. When this condition is satisfied we shall say that $(A, B_0, \cdots, B_{m-1})=(A, B)$ is *closeable elliptic.* We can now state

8. THEOREM. *Let (A, B) be closeable elliptic. Then a necessary and sufficient condition that the map $u\to(A-\lambda, B_0u, \cdots, B_{m-1}u)$ is a topological isomorphism of $H^{2m}(\Omega)$ onto $H^0(\Omega)x\prod_{j=0}^{m-1} H^{2m-m_j-1/2}\,\Gamma$ and that* (6.1) *be true is that $A(\xi)-\lambda\neq 0$ for $\xi\in\boldsymbol{R}^n$.*

9. DEFINITION. Let $V_B(\Omega)=\{u\in\mathfrak{D}(\bar\Omega): B_ju=0, j=0, \cdots, m-1\}$ and $V_B^{2m}(\Omega)$ be the closure of $V_B(\Omega)$ in $H^{2m}(\Omega)$. Let the domain of $A_B(\mathfrak{D}(A_B))$ be $V_B^{2m}(\Omega)$ and $A_Bu=Au$ for $u\in\mathfrak{D}(A_B)$.

10. DEFINITION. If T is a closed linear operator in a Banach space $\mathbf{X}$, the resolvent set of $T(\rho(T))$ is the set of $\lambda\in\boldsymbol{C}$: $T-\lambda I$ has a densely defined continuous inverse. The set $\boldsymbol{C}\sim\rho_T$ is called the spectrum of

[3] $A_\lambda{}^+(\xi', \tau)=\Pi_{j=1}(\tau-\tau_j(\xi'))$, $A^+=A_0^+$.

$T(\sigma(T))$. If $T-\lambda I$ has a densely defined but not a continuous inverse then λ is said to be in the continuous spectrum of T, $\sigma_c(T)$.

11. THEOREM. *Let* (A, B) *be closeable elliptic,* B *a normal set of boundary operators in the sense of Aronszajn-Milgram* [2]. *Then*

$$\rho(A_B) = \{\lambda \in \boldsymbol{C}: A(\xi) - \lambda \neq 0 \quad \text{for } \xi \in R^n\}.$$

If $\lambda \notin \rho(A_B)$ then $\lambda \in \sigma_C(A_B)$.

Using interpolation theory in Hilbert Space we have the result

12. THEOREM. *If* (A, B) *is closeable elliptic and* B *is normal, then for* $\lambda \in C$ *such that* $A(\xi) - \lambda \neq 0$, $\xi \in \boldsymbol{R}^n$, *the map* $u \rightarrow \{(A - \lambda)u, B_0u, \cdots, B_{m-1}u\}$ *is a topological isomorphism of the completion* $H^s(A - \lambda, \Omega)$ *of* $\mathfrak{D}(\bar{\Omega})$ *in the norm* $\|u\|_{A-\lambda,s} = (\|(A-\lambda)u\|_{s-2m}^2 + \|u\|_s^2)^{1/2}$ *onto* $H^{s-2m}(\Omega) x \prod_{j=0}^{m-1} H^{s-m_j-1/2}\ \Gamma$ *for* $0 \leqq s \leqq 2m$. *Here for* $u \in \mathfrak{D}(\bar{\Omega})$ $\|u\|_{-s} = \sup_{v \in H^s} |(u, v)| / \|v\|_s$ *for* $s > 0$. *The norms* $\|\cdot\|_s$ *for* s *real can be defined either by interpolation or by for* $u \in \mathfrak{D}(\bar{\Omega}) \|u\|_s = \inf\{\|\tilde{u}\|_s: \tilde{u} \in \mathfrak{D}$ *and* $\tilde{u}|_\Omega = u\}$. *Then* $H^s(\Omega)$ *is the completion of* H^s *in the norm so defined.*

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