ON A CHARACTERIZATION OF TYPE I C^*-ALGEBRAS[1]

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1. **Introduction.** Recently, establishing a conjecture of Calkin [1], the author [7] showed the following result: Let $\mathfrak{H}$ be a separable Hilbert space, $B(\mathfrak{H})$ the C^*-algebra of all bounded operators on $\mathfrak{H}$, $C(\mathfrak{H})$ the C^*-algebra of all compact operators on $\mathfrak{H}$, then the quotient algebra $B(\mathfrak{H})/C(\mathfrak{H})$ has a type III-factor $*$-representation. The discussions which are used in the proof of this result are applicable to more general situations. In the present paper, by using those discussions and the result of Glimm [3], we shall give a characterization of type I C^*-algebras without the assumption of separability as follows:

MAIN THEOREM. *Let A be a C^*-algebra. Then the following conditions are equivalent.*

(1) *A is a* GCR *algebra,*

(2) *A is of type* I,

(3) *A has no type* III*-factor $*$-representation.*

2. **Theorems.** First of all we shall state a generalization of the result which are crucial in the proof of Calkin's conjecture.

THEOREM 1. *Let A be a C^*-algebra with unit I, B a C^*-sub algebra containing I of A and M a type* III*-factor on a separable Hibert space. Suppose that there is a linear mapping P of A into M satisfying the following conditions:*

(1) $P(x^*)=P(x)^*$ *for* $x\in A$,

(2) $P(h)\geqq 0$ *for* $h(\geqq 0)\in A$,

(3) $P(axb)=P(a)P(x)P(b)$ *for* $a, b\in B$ *and* $x\in A$,

(4) *$P(B)$ is σ-weakly dense in M. Then, A has a type* III*-factor $*$-representation.*

The proof of this theorem is similar to the one in [7]. Here we shall sketch the proof. Let Ω be the set of all linear mappings Q of A into M satisfying the conditions (1), (2), (3) and $Q(a)=P(a)$ for $a\in B$.

Let $\mathfrak{L}(A, M)$ be the Banach space of all bounded linear mappings of A into M. Then it is the dual of a Banach space $A\otimes_\gamma M_*$, where M_* is the associated space (namely, the dual of $M_*=M$) and γ is the greatest cross norm.

LEMMA 1. *Ω is a $\sigma(\mathfrak{L}(A, M), A\otimes_\gamma M_*)$-compact convex subset of $\mathfrak{L}(A, M)$ and each $Q\in\Omega$ satisfies $Q(x^*x)\geqq Q(x)^*Q(x)$ for $x\in A$.*

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The first part of Lemma 1 is clear. For the last part, by the assumptions (1), (3), and (4), and the density theorem of Kaplansky, there is a direct set (a_α) in B such that $\|a_\alpha\| \leqq \|Q(x)\|$ and $Q(a_\alpha) \rightarrow Q(x)$ (strongly) in M. Then, for ϕ $(\geqq 0) \in M_*$,

$$\begin{aligned} \langle Q(x)^*Q(x), \phi\rangle &= \lim_\alpha \langle Q(x)^*Q(a_\alpha), \phi\rangle \\ &= \lim_\alpha \langle Q(x^*a_\alpha), \phi\rangle = \lim \langle x^*a_\alpha, Q^*(\phi)\rangle \\ &\leqq \lim_\alpha \sup \langle x^*x, Q^*(\phi)\rangle^{1/2}\langle a_\alpha^*a_\alpha, Q^*(\phi)\rangle^{1/2} \end{aligned}$$

(because $Q^*(\phi) \geqq 0$ by (2))

$$= \lim_\alpha \sup \langle Q(x^*x), \phi\rangle^{1/2}\langle Q(a_\alpha)^*Q(a_\alpha), \phi\rangle^{1/2}.$$

Hence $Q(x)^*Q(x) \leqq Q(x^*x)$ for $x \in A$. This completes the proof.

Let ϕ be a normal, faithful state on M. For $Q \in \Omega$, we shall define a state ϕ_Q on A by $\phi_Q(x) = \phi(Q(x))$ for $x \in A$. Let $\mathcal{E} = \{\phi_Q \mid Q \in \Omega\}$; then by Lemma 1 we can easily show that $\mathcal{E}$ is a compact convex subset of the state space of A. Let $\phi_Q(Q \in \Omega)$ be an extreme point of $\mathcal{E}$, and let $\{\pi_Q, \mathfrak{H}_Q\}$ be the $*$-representation of A on a Hilbert space $\mathfrak{H}_Q$ constructed via ϕ_Q. Then, we shall define a linear mapping of N onto M in the following, where N is the weak closure of $\pi_Q(A)$ on $\mathfrak{H}_Q$.

For $f \in M_*$, we define $F(\pi_Q(x)) = f(Q(x))$ for $x \in A$. This is well defined, because $\pi_Q(x) = 0$ implies $\phi_Q(x^*x) = \phi(Q(x^*x)) = 0$ and so $Q(x^*x) \geqq Q(x)^*Q(x) = 0$, so that $Q(x) = 0$. Then, F is strongly continuous on bounded spheres (cf. [7]) and so by Lemma 3 in [7], the F can be uniquely extended to an element $\overline{F}$ of N_*, where N_* is the associated space of N with $\|\overline{F}\| = \|F\|$.

Put $T(f) = \overline{F}$ for $f \in M_*$, then T is a bounded linear mapping of M_* into N_*. Let T^* be the dual of T, then T^* is a continuous linear mapping of N with the topology $\sigma(N, N_*)$ into M with the topology $\sigma(M, M_*)$.

LEMMA 2. *T^* satisfies the following conditions*:

(1) $T^*(\pi_Q(x)) = Q(x)$ *for* $x \in A$,

(2) $T^*(y^*) = T^*(y)^*$ *for* $y \in N$,

(3) $T^*(h) \geqq 0$ *for* $h(\geqq 0) \in N$,

(4) $T^*(uyv) = T^*(u)T^*(y)T^*(v)$ *for* u, $v \in$ *the σ-weak closure of* $\pi_Q(B)$ *in* N *and* $y \in N$,

(5) $T^*(y^*y) \geqq T^*(y)^*T^*(y)$ *for* $y \in N$.

The proof of this lemma is quite similar with the proof of Lemma 4 in [7].

LEMMA 3. *N is a factor.*

The proof is similar with the proof of Lemma 5 in [7]. Now we shall prove Theorem 1.

PROOF OF THEOREM 1. Let $[\pi_Q(B)I_Q]$ be the closed subspace of $\mathfrak{H}_Q$ generated by $\pi_Q(B)I_Q$, where I_Q is the image of I in $\mathfrak{H}_Q$ and E' be the projection of $\mathfrak{H}_Q$ onto $[\pi_Q(B)I_Q]$, then E' belongs to the commutant $\pi_Q(B)'$ of $\pi_Q(B)$.

$$\phi_Q(a^*bc) = \phi(Q(a^*bc)) = \phi(Q(a)^*Q(b)Q(c)) \quad \text{for } a, b, c \in B$$

and $Q(B)$ is σ-weakly dense in M, and so the $*$-isomorphism $\pi_Q(b)E' \to Q(b)$ of $\pi_Q(B)E'$ into M can be uniquely extended to a $*$-isomorphism ρ of a W^*-algebra $\pi_Q(B)''E'$ onto M; therefore $\pi_Q(B)''E'$ and $E'\pi_Q(B)'E'$ are type III-factors.

Let F' be the central envelope of E' in $\pi_Q(B)'$, then F' belongs to $\pi_Q(B)''$. The mapping η: $xF' \to xE'$ ($x \in \pi_Q(B)''$) of $\pi_Q(B)''F'$ onto $\pi_Q(B)''E'$ is a $*$-isomorphism. Therefore the mapping $\rho \cdot \eta$ of $\pi_Q(B)''F'$ onto M is a $*$-isomorphism.

Now suppose that N is semifinite, then there is a normal semifinite faithful trace τ on N.

Put $N_o = \{e \mid \tau(e) < +\infty,\ e \text{ projections in } N\}$. $F' \in \pi_Q(B)''$ and $T^*(F') \neq 0$, because $\langle T^*(F'), \phi \rangle = \langle F', T(\Phi_Q) \rangle = (F'I_Q, I_Q)$ and $I_Q \in E'\mathfrak{H}_Q$, where (,) is the inner product of $\mathfrak{H}_Q$.

Therefore, there is a nonzero projection $e_o \in N_o$ such that $e_o \leqq F'$ and $T^*(e_o) \neq 0$, and so there is a nonzero projection p in M such that $\lambda p \leqq T^*(e_o)$ for some positive number λ.

Suppose that a directed set (a_α) ($\|a_\alpha\| \leqq 1$, $a_\alpha \in pMp$) converges strongly to 0 in M, then $\{\eta^{-1}\rho^{-1}(a_\alpha)\}$ converges strongly to 0 on $\mathfrak{H}_Q$ and so $\{\eta^{-1}\rho^{-1}(a_\alpha)e_o\}$ converges strongly to 0 on $\mathfrak{H}_Q$; by the finiteness of e_o, $\{e_o(\eta^{-1}\rho^{-1}(a_\alpha))^*\}$ converges strongly to 0 (cf. [5], [6]). Then,

$$T^*((e_o(\eta^{-1}\rho^{-1}(a_\alpha))^*)^*(e_o(\eta^{-1}\rho^{-1}(a_\alpha))^*))$$
$$\geqq T^*(e_o(\eta^{-1}\rho^{-1}(a_\alpha))^*)^*T^*(e_o(\eta^{-1}\rho^{-1}(a_\alpha))^*) \to 0 \ (\sigma\text{-weakly}) \text{ in } M;$$

hence $\{T^*(e_o(\eta^{-1}\rho^{-1}(a_\alpha))^*)\}$ converges strongly to 0 in M.

For $a \in M$, we choose a bounded directed set $\{Q(b_\beta)\}$ such that $b_\beta \in B$ and $Q(b_\beta) \to a$ (σ-weakly) in M, then $\eta^{-1}\rho^{-1}(Q(b_\beta)) = \pi_Q(b_\beta)F' \to \eta^{-1}\rho^{-1}(a)$ (σ-weakly) in $\pi_Q(B)''$; moreover, $T^*(\eta^{-1}\rho^{-1}(Q(b_\beta)) = T^*(\pi_Q(b_\beta)F') = T^*(\pi_Q(b_\beta))T^*(F') = Q(b_\beta)T^*(F') \to aT^*(F')$ (σ-weakly) in M; hence $aT^*(F') = T^*(\eta^{-1}\rho^{-1}(a))$ for all $a \in M$.

Therefore,

$$\{pT^*(e_o)p + (I - p)\}^{-1}pT^*(e_o(\eta^{-1}\rho^{-1}(a_\alpha))^*)$$
$$= \{pT^*(e_o)p + (I - p)\}^{-1}pT^*(e_o)T^*(\eta^{-1}\rho^{-1}(a_\alpha))^*$$

(because $\eta^{-1}\rho^{-1}(a_\alpha)\in\pi_Q(B)''$, and $T^*(e_o)T^*(F')=T^*(e_oF')=T^*(e_o)$) $=\{pT^*(e_o)p+(I-p)\}^{-1}pT^*(e_o)a_\alpha^*=a_\alpha^*\to 0$ (strongly) in M.

Hence, the $*$-operation is strongly continuous on bounded spheres of pMp, but pMp is of type III. This is a contradiction (cf. [5], [6]). This completes the proof.

Now we shall show

THEOREM 2. *Let A be a C^*-algebra. Then the following conditions are equivalent*:

(1) *A is* GCR,

(2) *A is of type* I,

(3) *A has no type* III-*factor $*$-representation.*

PROOF. (1)$\Rightarrow$(2) is Theorem 6 of Kaplansky [4]. (2)$\Rightarrow$(3) is clear from the definition of type I C^*-algebras. Now we shall show that (3)$\Rightarrow$(1). Suppose that A is not GCR. Let $\mathfrak{b}$ be the maximum GCR ideal of A (cf. [4]), then the quotient algebra $A/\mathfrak{b}$ has no nonzero GCR ideal. If we can show that $A/\mathfrak{b}$ has a type III-factor $*$-representation, then A has it: therefore we can assume that $\mathfrak{b}=(0)$ and moreover A has unit I.

Then by the results of Glimm (Lemmas 4 and 5 and the proof of (b1)$\Rightarrow$(b2); (b1)$\Rightarrow$(b3) of Theorem 1 in [3]), A contains a nontype I separable C^*-subalgebra B.

Then by the results of Glimm (pp. 588–589, [3]) and Schwartz [9], B has a type III-factor $*$-representation $\{\pi, \mathfrak{H}\}$ on a separable Hilbert space $\mathfrak{H}$ such that $\pi(B)'$ has the property P in the sense of Schwartz and so there is a linear mapping R of the C^*-algebra $B(\mathfrak{H})$ of all bounded operators on $\mathfrak{H}$ onto $\pi(B)''$ satisfying the conditions (1) $R(x^*)=R(x)^*$ for $x\in B(\mathfrak{H})$, (2) $R(h)\geqq 0$ for h $(\geqq 0)\in B(\mathfrak{H})$, (3) $R(axb)=aR(x)b$ for a, $b\in\pi(B)''$ and $x\in B(\mathfrak{H})$, and $R(I)=I$.

Now let $\xi(\|\xi\|=1)$ be a separating and generating vector of $\pi(B)''$ (cf. [2]) and put $\chi(a)=(\pi(a)\xi, \xi)$ for $a\in B$.

Let $\tilde{\chi}$ be an extended state of χ on A and let $\{\pi_\chi, \mathfrak{H}_\chi\}$ be the $*$-representation of A constructed via $\tilde{\chi}$.

Let $[\pi_{\tilde{\chi}}(B)I_{\tilde{\chi}}]$ be the closed subspace of $\mathfrak{H}_{\tilde{\chi}}$ generated by $\pi_{\tilde{\chi}}(B)I_{\tilde{\chi}}$ and E' be the projection of $\mathfrak{H}_{\tilde{\chi}}$ onto $[\pi_{\tilde{\chi}}(B)I_{\tilde{\chi}}]$, then the representation $b\to\pi(b)$ of B can be canonically identified with the representation $b\to\pi_{\tilde{\chi}}(b)E'$ of B. Then R is a linear mapping of $B(E'\mathfrak{H}_{\tilde{\chi}})$ onto $\pi_{\tilde{\chi}}(B)''E'$.

Now we shall define a linear mapping P of $\pi_{\tilde{\chi}}(A)$ into the type III-factor $\pi_{\tilde{\chi}}(B)''E'$ as follows: $P(\pi_{\tilde{\chi}}(x))=R(E'\pi_{\tilde{\chi}}(x)E')$ for $x\in A$. Then, we can easily show that P satisfies the conditions of Theorem 1.

Hence the C^*-algebra $\pi_{\chi}(A)$ and so A have a type III-factor $*$-representation.

This completes the proof.

References

1. J. W. Calkin, *Two sided ideals and congruences in the ring of bounded operators in Hilbert space*, Ann. of Math. **42** (1941), 839–873.

2. J. Dixmier, *Les algebres d'operateurs dans l'espace hilbertien*, Gauthier-Villars, Paris, 1957.

3. J. Glimm, *Type* I *C*-algebras*, Ann. of Math. **73** (1961), 572–612.

4. I. Kaplansky, *The structure of certain operator algebras*, Trans. Amer. Math. Soc. **70** (1951), 219–255.

5. S. Sakai, *On topological properties of* W**-algebras*, Proc. Japan Acad. **33** (1957), 439–444.

6. ———, *The theory of W*-algebras*, Lecture notes, Yale University, 1962.

7. ———, *On a problem of Calkin*, Amer. J. Math. (to appear).

8. J. Schwartz, *Two finite, non-hyperfinite, nonisomorphic factors*, Comm. Pure Appl. Math. **16** (1963), 19–26.

9. ———, *Non-isomorphism of a pair of factors of type* III, Comm. Pure Appl. Math. **16** (1963), 111–120.

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SOME UNSYMMETRIC COMBINATORIAL NUMBERS

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By an *n-configuration* we shall mean an abstract set of n elements, together with the set of all unordered pairs of distinct elements from the set. It is convenient also to use quasi-geometrical terminology such as *vertex* for element, *edge* or *side* for a pair (2-tuple), *triangle* as well as triple (3-tuple) for a 3-subconfiguration, and so on.

The Ramsey number $N(p, q, 2)$ (see [**3**, pp. 38–43], or [**2**, pp. 61–65]), for two kinds h, v of pairs (or two "colors of edges"), is the smallest integer such that if $n \geqq N(p, q, 2)$, then any n-configuration is sure to contain *either* an h p-tuple (a p-tuple all of whose edges are h) *or* a v q-tuple. Call a p-tuple all of whose edges are alike (h or v) a *like* p-tuple. We introduce, and partially determine the values of, new analogous combinatorial numbers $K(p, q, 2)$, $M(p, q, 2)$, and $V(p, q, 2)$.

Definitions. The number $K(p, q, 2)$ is the smallest integer such that if $n \geqq K(p, q, 2)$, then for each vertex, the configuration is sure to contain *either* a like p-tuple containing the vertex, *or* a like q-tuple not containing the vertex. For three kinds r, g, v of edges, $M(p, q, 2)$