SOLVABILITY OF THE FIRST COUSIN PROBLEM AND VANISHING OF HIGHER COHOMOLOGY GROUPS FOR DOMAINS WHICH ARE NOT DOMAINS OF HOLOMORPHY. II

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This work is a continuation of [2]. In [2] we studied the cohomology groups $H^q(X\backslash A, \mathcal{O})$ where $A(\subset X)$ is a closed generalized polydisc. Here we consider the general case where A is the closure of a domain of holomorphy. This general case was treated in [1] for $q=1$, but the present method (for $q\geqq 1$) is entirely different.

We adopt the definition in [4] of analytic polyhedron. By an analytic polyhedron in *general position* we mean an analytic polyhedron as defined in [3, p. 288].

THEOREM 1. *Let* $A\subset \mathbf{C}^n$ *be the closure of a bounded analytic polyhedron in general position and let* X *be any open set in* $\mathbf{C}^n$, *containing* A. *Then the restriction map*

$$H^q(X, \mathcal{O}) \to H^q(X\backslash A, \mathcal{O}) \qquad (1 \leqq q \leqq n-2) \tag{1}$$

is bijective.

We proceed as in [2] except that now we take $G=B\backslash A$ where $B=\{z\in D;\ f_j(z)\in\Delta_j' \text{ for } j=1, \cdots, N\}$ where A is defined by $A=\{z\in D; f_j(z)\in\Delta_j \text{ for } j=1, \cdots, N\}$ where f_j are holomorphic in D, Δ_j' is some open neighborhood of $\bar{\Delta}_j$, and $\bar{B}\subset D$. (The argument in [2] can be simplified by dropping out the sets $U_{i_1}, \cdots, U_{i_q}$ which occur in the covering $X\backslash A$.) All we need to prove is the following lemma.

LEMMA. $H^p(G, \mathcal{O})=0$ *for* $1\leqq p\leqq n-2$.

PROOF. For simplicity we take Δ_j to be the unit disc and Δ_j' to be a disc with radius $1+\epsilon$, homothetic to Δ_j. Clearly $G=\bigcup_{i=1}^N U_i$ where U_i is defined as B except for the additional condition $|f_i(z)|>1$. Thus, each U_i is also an analytic polyhedron. We next proceed analogously to [6, p. 349] and represent $f_{i_0\cdots i_p}$ in $U=\bigcap_{i=1}^N U_i$ as $\sum C_M(f_{i_0\cdots i_p})$ where $M=\{M', M''\}$ is a set of indices $j_1, \cdots, j_n$ such that the integration in $C_M(f)$ is taken over $|f_{j_1}|=\gamma_1, \cdots, |f_{j_n}|=\gamma_n$ where $\gamma_h=1$ if $j_h\in M''$ and $\gamma_h=1+\epsilon$ if $j_h\in M'$; the above integral representation is that given by the Cauchy-Weil formula [3],

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[7]. Actually we should have used the representation in compact subsets of U, but since this does not affect all the arguments below, we simplify the notation by representing $f_{i_0\cdots i_p}$ in U.

One verifies that (α) if $i\in M''$, $i\notin\{i_0, \cdots, i_p\}$ then $C_M(f_{i_0\cdots i_p})=0$. Indeed this follows by the Cauchy-Poincaré theorem [3, p. 264] applied in the $(n+1)$-dimensional set defined by $|f_j|=1+\epsilon$ for $j\in M'$, $|f_j|=1$ for $j\in M''\setminus\{i\}$, and $|f_i|\leqq 1$. Since $p+1<n$, it follows from (α) that (β) $C_M(f_{i_0\cdots i_p})=0$ if $M''=\{1, 2, \cdots, n\}$. Next, (γ) $C_M(f_{i_0\cdots i_p})$ is holomorphic in $U_{i_0\cdots i_p}=\bigcap_{j=0}^{p} U_{i_j}$, since, by ($\alpha$), we may assume that $M''\subset\{i_0, \cdots, i_p\}$. Finally, ($\delta$) if $i\notin M''$ then $C_M(f_{ii_0\cdots i_{p-1}})$ is holomorphic in $U_{i_0\cdots i_{p-1}}$. To construct g with $\delta g=f$ (f any given p-cocycle) it suffices to construct, for each fixed M, g with $\delta g=C_M(f)$. We may assume that there is an $i\notin M''$, $1\leqq i\leqq n$, since otherwise $C_M(f)=0$ by (β). We then take $g_{i_0\cdots i_{p-1}}=C_M(f_{ii_0\cdots i_{p-1}})$.

COROLLARY 1. *If $N=n$ in Theorem 1 then there is a surjective map* (1) *also for $q=n$. If, further, X is a domain of holomorphy, then* $H^{n-1}(X\setminus A, \mathfrak{O})\neq 0$.

The first part follows by observing that $H^n(G, \mathfrak{O})=0$ and using the Mayer-Vietoris sequence (see [0, p. 236]) for B, $X\setminus A$. If the second part is false then $H^q(X\setminus A, \mathfrak{O})=0$ for $1\leqq q\leqq n$. Employing Dolbeault's theorem and [5, Theorem 4.2.9] it follows that $X\setminus A$ is a domain of holomorphy.

THEOREM 2. *Let $A=\bigcap_{j=1}^{\infty} X_j$ where $X_{j-1}\supset\overline{X}_j$, X_j is a bounded domain of holomorphy in $\mathbf{C}^n$ and A is a closed set, and let X be any open set in $\mathbf{C}^n$, containing A. Then the restriction map* (1) *is bijective for $1\leqq q\leqq n-2$ and injective for $q=n$.*

PROOF. Each X_j can be exhausted by a sequence of analytic polyhedra with $N=n$ (see [4, p. 218]), and by slightly modifying the domains in which the values of the functions (defining the analytic polyhedron) lie, we get a sequence of analytic polyhedra in general position. Thus we can write $A=\bigcap_{j=1}^{\infty} P_j$ where P_j are analytic polyhedra in general position, and $P_{j-1}\supset\overline{P}_j$. Take a covering W of $X\setminus A$ by domains of holomorphy such that for each $j=1, 2, \cdots$, there is a subset of W which is a covering of $X\setminus\overline{P}_j$ and such that the closure of each set in W does not intersect ∂A. Using Leray's lemma [4] and the fact that the restriction maps

$$H^r(X, \mathfrak{O})\rightarrow H^r(X\setminus\overline{P}_j, \mathfrak{O})\qquad (r=q, q-1)\tag{2}$$

are bijective (for any $1\leqq q\leqq n-2$) it follows by the isomorphisms $H^r(X\setminus\overline{P}_j, \mathfrak{O})\rightarrow(H^r(X\setminus P_{j-1}\mathfrak{O}))$ and [0, p. 241 and p. 250] that the map (1) is bijective we actually need only the injectivity of (2) for $r=q$ and the surjectivity of (2) for $r=q, q-1$.

EXAMPLE. If A is a compact convex set, or if ∂A is C^2 and strictly pseudoconvex, then A satisfies the assumptions in Theorem 2. If, in particular, X is a domain of holomorphy, then $H^q(X\backslash A, \mathcal{O})=0$ for $1\leqq q\leqq n-2$.

Added in proof. Theorems 1, 2 remain true if $\mathcal{O}$ is replaced by any coherent analytic sheaf $\mathfrak{F}$ over x, free in a neighborhood of A. Assume now that $\mathfrak{F}$ has a free resolution of length d in a neighborhood of ∂A. Then the lemma holds for $1\leqq p\leqq n-2-d$. Using a covering of $x\backslash A$ as in [2] and, additionally, a domain of holomorphy U_* containing A but not intersecting the U_j for $j=i_0, \cdots, i_q$, we get:

THEOREM 3. *If A, X are as in Theorem 2 and if $\mathfrak{F}$ is as above, then the restriction map* (1), *with $\mathcal{O}$ replaced by $\mathfrak{F}$, is bijective for $2\leqq q\leqq n-2-d$.*

This theorem yields the following result on cohomology with compact support: $H^q_0(\Omega, \mathfrak{F})=0$ *for* $2\leqq q\leqq n-1-d$, *if Ω is a domain of holomorphy in C^n.* (Overlapping results were proved, by a different method, in [0], using Serre's duality theorem.)

PROOF FOR $\mathfrak{F}=\mathcal{O}$: Given a $\bar\partial$-closed q-form f, with compact support in Ω, solve $\bar\partial g=f$ in C^n; then solve $\bar\partial v=g$ outside some compact analytic polyhedron in Ω (using Theorem 3). $u=v-\bar\partial(\zeta v)$, for some $\zeta\epsilon C_0^\infty$ satisfies $\bar\partial u=f$ and has compact support in Ω. For general $\mathfrak{F}$ we work with q-cochains and coboundary operators. The above proof, together with $H^n_0(\Omega, \mathcal{O})\neq 0$, leads to:

COROLLARY. *If Ω is a domain of holomorphy and a star domain, and if B is any open set with $\Omega\subset\subset G$, then $H^{n-1}(B\backslash\bar\Omega, \mathcal{O})\neq 0$.*

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