A CHARACTERIZATION OF Q-DOMAINS

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Let R be an integral domain with quotient field K. By an *overring* of R is meant a ring B with $R \subseteq B \subseteq K$. R is a *Q-domain* if every overring of R is a ring of quotients of R with respect to some multiplicative system in R. A *P-domain* is a Prüfer ring. Q-domains have been investigated by Gilmer and Ohm [3] and by Davis [2]. All Q-domains are P-domains, and a long list of characterizations of P-domains is available in Bourbaki [1, pp. 93–94]. Noetherian Q-domains are characterized in [3] as those Dedekind domains whose ideal class group is a torsion group. The purpose of this paper is to obtain a characterization of general Q-domains (Theorem 5).

Let K^* denote the set of nonzero elements of K. If $x \in K^*$, we define the *numerator ideal* of x to be $N(x) = \{a \in R: a = bx, \text{ for some } b \in R\}$ and the *denominator ideal* of x to be $D(x) = \{b \in R: bx \in R\}$. Since $N(x) = Rx \cap R$ and $D(x) = N(1/x)$, $N(x)$ and $D(x)$ are ideals in R.

If P is a prime ideal in R, R_P denotes the local ring of R at P.

THEOREM 1. *R is a P-domain if and only if $N(x) + D(x) = R$, for all $x \in K^*$.*

PROOF. First note that for any prime ideal $P \subseteq R$, $x \in R_P$ if and only if $D(x) \not\subseteq P$, and hence $1/x \in R_P$ if and only if $N(x) \not\subseteq P$. Therefore R_P is a valuation ring if and only if $N(x) \not\subseteq P$ or $D(x) \not\subseteq P$, for all $x \in K^*$, i.e., if and only if $N(x) + D(x) \not\subseteq P$, for all $x \in K^*$. Thus to say the ideals $N(x) + D(x)$ are all improper is equivalent to saying all the local rings R_P are valuation rings, i.e., R is a P-domain.

COROLLARY 2. *If R is a P-domain and $x \in K^*$, then the numerator and denominator ideals of x can be generated by two elements.*

PROOF. Since $N(x) + D(x) = R$ we can write $x = a/b = a'/b'$, where $a + b' = 1$. Then $D(x) = (b, b')$, for $c \in D(x)$ implies $c = ca + cb' = cxb + cb'$, with $cx \in R$. Also $N(x) = D(1/x) = (a, a')$.

In order to prove Theorem 5, we need to make two remarks concerning P-domains.

REMARK 3. If R is a P-domain, then the finitely generated fractionary ideals of R form a group [1]. Moreover, if $A = (a_1, \cdots, a_n)$

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is a finitely generated ideal, then the inverse of A is given by $A^{-1}=R:A=\{x\in K \mid xA\subseteq R\}$. Equivalently, $A^{-1}=(b_1, \cdots, b_n)$, where $a_ib_j\in R$ and $\sum a_ib_i=1$ [4, pp. 271–272].

REMARK 4. If R is a P-domain, and B is any overring of R, then B is the intersection of all the local rings R_P of R which contain B (Proposition 2 of [2]).

THEOREM 5. *Let R be a P-domain. Then R is a Q-domain if and only if for every finitely generated ideal $A\subseteq R$, there is an element $f\in R$ such that $\sqrt{A}=\sqrt{(f)}$.*

PROOF. First let R be a Q-domain. We follow the argument of Theorem 2.5(g) of [3]. Let $A=(a_1, \cdots, a_n)$ be a finitely generated ideal in R. Let B be the A-transform of R, i.e., $B=\{x\in K \mid xA^n\subseteq R$ for some $n\geqq 0\}$. By Remark 3, we may write $A^{-1}=(b_1, \cdots, b_n)$ with $\sum a_ib_i=1$, and then $B=\bigcup (A^n)^{-1}=\bigcup (A^{-1})^n=R[b_1, \cdots, b_n]$. Now B is a ring of quotients of R with respect to some multiplicative system S. Thus there is $f\in S$ such that $b_i=c_i/f$, $1\leqq i\leqq n$, with $c_i\in R$. Then $f=f(\sum a_ib_i)=\sum c_ia_i\in A$. Moreover, $f\in S$ implies $1/f\in B$, so $(1/f)(A^n)\subseteq R$, for some n, i.e., $A^n\subseteq (f)$. Since $A^n\subseteq (f)\subseteq A$, it follows that $\sqrt{A}=\sqrt{(f)}$.

Conversely suppose R satisfies the condition stated in the theorem. To prove R is a Q-domain it suffices to prove $R[x]$ is a ring of quotients of R, for every $x\in K^*$, by Proposition 1.4 of [3]. Let $x\in K^*$. Then, by Corollary 2, $D(x)$ is finitely generated, so $\sqrt{D(x)}=\sqrt{(f)}$, for some $f\in R$. Hence if P is any prime ideal in R we have $(f)\subseteq P$ if and only if $D(x)\subseteq P$, and thus $R[1/f]\subseteq R_P\Leftrightarrow 1/f\in R_P\Leftrightarrow f\notin P\Leftrightarrow D(x)\nsubseteq P\Leftrightarrow x\in R_P\Leftrightarrow R[x]\subseteq R_P$. It follows by Remark 4 that $R[x]=R[1/f]$ which is a ring of quotients of R. This completes the proof of Theorem 5.

It should be noted that Theorem 5 does not answer the question raised on [3], namely: is the ideal class group of every Q-domain a torsion group?

REFERENCES

1. N. Bourbaki, *Algèbre commutative*, Hermann, Paris, 1965, Chapitre 7.

2. E. D. Davis, *Overrings of commutative rings*. II, Trans. Amer. Math. Soc. **110** (1964), 196–212.

3. R. Gilmer and J. Ohm, *Integral domains with quotient overrings*, Math. Ann. **153** (1964), 97–103.

4. O. Zariski and P. Samuel, *Commutative algebra*, Vol. I, Van Nostrand, Princeton, N. J., 1958.

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