SINGULAR INTEGRALS

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In this paper I will attempt to describe the subject as it has developed in the last fifteen years, outlining the methods by which its problems have been approached and discussing its connections with other branches of Analysis. It will perhaps be best to start by considering certain classical situations which lead naturally to singular integrals and which contain the seeds of some of the methods and some of the applications we will discuss later. The simplest one arises in attempting to establish the connection between the real and imaginary parts of the boundary values of an analytic function. Suppose that $f(z)$ is analytic in $I(z) \geqq 0$ and that $zf(z)$ is bounded, then if $u(x)$ and $v(x)$ are the real and imaginary parts of f on $R(z)=0$, we have

$$f(z) = \frac{1}{\pi i}\int_{-\infty}^{\infty} \frac{u(t)}{t - z}\, dt, \qquad v(x) = \lim_{\epsilon\to 0} \frac{1}{\pi}\int_{|x-t|>\epsilon} \frac{u(t)}{x - t}\, dt$$

the second formula being obtained from the first by letting the imaginary part of z tend to zero. The expression for v above is the so-called Hilbert transform of the function u. The integral it involves is absolutely divergent due to the singularity of the kernel at $x=t$.

A second example is given by the second order derivatives of the Newtonian potential. Let us consider the potential of the density function $f(x)$ in R^3

$$g(x) = \frac{-1}{4\pi}\int \frac{f(y)}{|x - y|}\, dy,$$

where $|x-y|$ denotes the distance between the points x and y and dy denotes the volume element in R^3. If f is sufficiently smooth, differentiation under the integral sign leads to the following expression for the second derivatives $D_{ij}g$ of g

$$D_{ij}g(x) = \frac{-1}{4\pi}\int \frac{1}{|x - y|} D_{ij}f(y)\, dy$$

and integration by parts gives

$$D_{ij}g(x) = \tfrac{1}{3}\delta_{ij}f(x) + \lim_{\epsilon\to 0}\int_{|x-y|>\epsilon} k_{ij}(x - y)f(y)\, dy, \tag{1}$$

[1] Colloquium Lectures given from August 31 to September 3, 1965 at the Seventieth Summer Meeting of the American Mathematical Society held in Ithaca, New York.

where

$$4\pi k_{ij}(x) = (\delta_{ij} | x |^2 - 3x_i x_j) | x |^{-5}.$$

The preceding integral is analogous to the integral defining the Hilbert transform. In fact, its kernel $k_{ij}(x-y)$ is a homogeneous function of $x-y$ of degree equal to minus the number of variables on which f depends. Thus the kernel has a singularity at $x=y$ which makes the integral absolutely divergent. Nevertheless, if f is sufficiently smooth, the integral can still be defined as a principal value, as is readily seen from the expression for $D_{ij}g$ above. This possibility stems from the fact that the kernels $k_{ij}(x)$ have mean value 0 on the sphere $|x| = 1$, which produces enough cancellation to ensure the existence of the principal value. This property is shared by the kernel of the Hilbert transform and also accounts for its existence. Let us introduce now the following abbreviation

$$K_{ij}f = \lim_{\epsilon \to 0} \int_{|x-y|>\epsilon} k_{ij}(x - y) f(y) \, dy.$$

Then observing that $f = \Delta g$ we can rewrite (1) as follows

$$D_{ij}g = (\tfrac{1}{3}\delta_{ij} + K_{ij})(\Delta g) \tag{2}$$

and we find that the monomial differential operator D_{ij} can be expressed in terms of the Laplacian and the singular integral operator $\frac{1}{3}\delta_{ij} + K_{ij}$. This is not accidental. As we shall see later there is an intimate relation between differential operators and singular integral operators. This kinship is significant from the point of view of the fine local properties of functions as well as from the broader and more formal point of view of operator theory. The study of certain aspects of differential operators seems to lead inevitably to the consideration of singular integral operators. Let us now express the relationship (2) in terms of Fourier transforms. Since $(D_{ij}g)\hat{} = -4\pi^2 x_i x_j \hat{g}$ and $(\Delta g)\hat{} = -4\pi^2 |x|^2 \hat{g}$ we have

$$(D_{ij}g)\hat{} = x_i x_j | x |^{-2} (\Delta g)\hat{}, \qquad [(\tfrac{1}{3}\delta_{ij} + K_{ij})f]\hat{} = x_i x_j | x |^{-2} \hat{f}$$

and we see that in terms of Fourier transforms the operator $\frac{1}{3}\delta_{ij} + K_{ij}$ amounts simply to multiplication by the bounded function $x_i x_j |x|^{-2}$. This shows that our operators are bounded on L^2 and makes their formal properties transparent. We have thus two alternative ways of viewing our operators. The first one has turned out to be appropriate for the discussion of questions concerning existence, continuity etc., and the second is relevant to the formal aspects of the theory. These two examples will suffice as an introduction.

1. Translation invariant singular integral operators. Let $k(x)$ be a function defined in R^n-0, which is locally integrable and positively homogeneous of degree $-n$. Suppose in addition that $k(x)$ has mean value 0 on the sphere $|x|=1$. Let $k_\epsilon(x)=k(x)$ if $|x|\geqq\epsilon$ and $k(x)=0$ otherwise, and consider the following operator

$$(3)\qquad Kf = \lim_{\epsilon\to 0} K_\epsilon f, \qquad K_\epsilon f = \int_{R^n} k_\epsilon(x-y)f(y)\,dy.$$

It is not difficult to see that if f is sufficiently smooth and, say, has compact support the preceding limit exists. In fact, if $\phi(x)$ is any spherically symmetric bounded function with compact support, then the integral of $k_\epsilon(x)\phi(x)$ vanishes and therefore, if $\phi(x)=1$ near $x=0$ and f is differentiable

$$\begin{aligned} Kf &= \lim_{\epsilon\to 0}\int k_\epsilon(x-y)f(y)\,dy \\ &= \lim_{\epsilon\to 0}\int k_\epsilon(x-y)[f(y)-f(x)\phi(x-y)]\,dy \\ &= \int k(x-y)[f(y)-f(x)\phi(x-y)]\,dy \end{aligned}$$

the last integral being absolutely convergent. If $|k|\ \log^+|k|$ is integrable on $|x|=1$ one can show that the Fourier transform $\hat{k}_\epsilon(x)$ of $k_\epsilon(x)$ is bounded uniformly in ϵ and converges to a limit $\hat{k}(x)$ as ϵ tends to 0. Thus if f is square integrable $(K_\epsilon f)^\wedge=\hat{k}_\epsilon\hat{f}$ converges in the mean to $\hat{k}\hat{f}$ and $K_\epsilon f$ converges in the mean to a function whose Fourier transform is $\hat{k}\hat{f}$. If $f\in L^2$ the operator K is then well defined as a limit in the mean and is bounded on L^2. The function $\hat{k}(x)$ is the Fourier transform of the distribution $\lim_{\epsilon\to 0} k_\epsilon(x)$ and is homogenous of degree 0.

Let us consider now a slightly more general operator $Hf=cf+Kf$ where c is a constant. Then $(Hf)^\wedge=(c+\hat{k})\hat{f}=\hat{h}\hat{f}$ and $\hat{h}$ is again a homogeneous function of degree 0. Conversely, every sufficiently smooth homogeneous function $\hat{h}$ of degree 0 arises this way. For example, if $k(x)$ is infinitely differentiable so is $\hat{h}$, and every infinitely differentiable function $\hat{h}$ of degree zero corresponds to an operator H with infinitely differentiable kernel k. Thus the operators H with infinitely differentiable kernel are in one-to-one linear correspondence with the infinitely differentiable homogeneous functions $\hat{h}$ of degree 0. The function $\hat{h}$ is called the symbol of the operator H and is also denoted by $\sigma(H)$. It thus becomes clear that this class of operators is closed under composition and that

$$\sigma(H_1H_2) = \sigma(H_1)\sigma(H_2), \qquad \sigma(H^*) = \sigma(H^-),$$

that is, σ is a star-representation of the algebra of operators H with infinitely differentiable kernel into the algebra of infinitely differentiable homogeneous functions of degree 0. There are besides many other algebras of singular integral operators obtained by imposing various conditions on the kernels of k. For example, if we consider operators H for which $k(x)$ belongs to L^p, $1<p<\infty$, on $|x|=1$, we obtain an algebra of operators which is a Banach algebra with respect to the norm

$$|c| + \left[\int_{|x|=1} |k(x)|^p d\sigma\right]^{1/p},$$

where σ denotes the surface area of the sphere $|x|=1$, and whose representative function space also consists of homogeneous functions of degree 0.

Let us turn now to the connection between singular integral operators and differential operators. For this purpose we will introduce a fractional differentiation operator Λ which is defined as follows. Let f be square integrable and have square integrable first order derivatives. Then

$$(4) \qquad (\Lambda f)^\wedge = |x|\hat{f}.$$

As readily seen Λ maps the space L^2_k of square integrable functions with square integrable derivatives up to order k, continuously into the space L^2_{k-1}. Furthermore, $4\pi^2\Lambda^2 = -\Delta$ and

$$\left(\frac{1}{2\pi i}\frac{\partial}{\partial x_j}f\right)^\wedge = x_j\hat{f} = x_j|x|^{-1}(\Lambda f)^\wedge;$$

consequently

$$(5) \qquad \frac{1}{2\pi i}\frac{\partial}{\partial x_j}f = R_j\Lambda f, \qquad \Lambda f = \sum_1^n \frac{1}{2\pi i}\frac{\partial}{\partial x_j}R_j,$$

where R_j is the singular integral operator defined by $\sigma(R_j) = x_j|x|^{-1}$. More generally, if D is a differential operator with constant coefficients of homogeneous order m, that is, in which only derivatives of order m appear, and $P(x)$ is its characteristic polynomial, then

$$Df = H\Lambda^m f,$$

where H is a singular integral operator with $\sigma(H) = (2\pi i)^m P(x)|x|^{-m}$. This decomposition of D is of particular interest for the following reason. Differential operators are normally discontinuous and cannot be defined everywhere in function spaces with desirable metric properties. Singular integral operators on the other hand are not only continuous in L^2 but in a large class of function spaces as we shall see below. Furthermore, H carries the algebraic properties of D, and the undesirable ones are relegated to the factor Λ^m.

2. **Existence and continuity of translation invariant singular integrals.** As was pointed out above, there is a large variety of function spaces on which the singular integrals are defined and operate continuously. Essentially two methods have been employed so far to obtain these spaces. One is direct and is based on the Plancherel theorem and an extension to several variables of a well-known lemma of F. Riesz. The other consists in a reduction to the one-dimensional case, that is, to the Hilbert transform, to which the theory of analytic functions can be applied. Though some results can be obtained by either method, their scopes are different and deserve independent attention.

Let us discuss first the reduction to the Hilbert transform. For this purpose we will employ the following notation: If f is a function on the real line we will write

$$H_\epsilon f = \frac{1}{\pi}\int_{|x-t|>\epsilon} \frac{f(t)}{x-t}\,dt, \qquad Hf = \lim_{\epsilon\to 0} H_\epsilon f.$$

These operators also act on functions in R^n in the following fashion. Given a unit vector ν and a function $f(x)$ in R^n we let $H_\epsilon(\nu)f$ be the result of operating with H_ϵ and H respectively on the restrictions of f to lines parallel to the unit vector ν, or more explicitly

$$H_\epsilon(\nu)f = \frac{1}{\pi}\int_{|t|>\epsilon} \frac{f(x-t\nu)}{t}\,dt, \qquad H(\nu)f = \lim_{\epsilon\to 0} H_\epsilon(\nu)f.$$

Let now K and K_ϵ be as in (3) with k an odd function of x, that is, such that $k(x) = -k(-x)$. Then, integration in polar coordinates shows that if $f(x)$ is bounded and has compact support

$$K_\epsilon f = \frac{\pi}{2}\int k(\nu)H_\epsilon(\nu)f\,d\sigma \tag{6}$$

where $d\sigma$ denotes the surface area of the unit sphere in R^n. Thus, if B is a Banach space of functions containing the space S of infinitely differentiable, rapidly decreasing functions, and the norm N of B is rotation and translation invariant, and furthermore, N is continuous on S and $H_\epsilon(\nu)$ is continuous with respect to N uniformly in ϵ, then the same is true of K_ϵ and for $f \in C_0^\infty(R^n)$

$$N(K_\epsilon f) \leqq cN(f) \int | k(\nu) | \, d\sigma$$

with c independent of ϵ and k. Furthermore, if $H_\epsilon(\nu)f$ converges with respect to N as ϵ tends to zero, the same holds for $K_\epsilon f$.

To obtain the corresponding results in the case when the kernel $k(x)$ is even, one uses the fact that an operator with even kernel is a finite sum of products of operators with odd kernels. This can be readily surmised by observing that the symbol of an operator has the same parity as its kernel. The details of this argument are rather technical and for this reason we will just mention the final result. Suppose that N is a norm as above. Then if the kernel of K is such that $|k| \log^+|k|$ is integrable on $|x| = 1$, there exists a constant depending only on k and N such that $N(K_\epsilon f) \leqq cN(f)$. The condition that $|k| \log^+|k|$ be integrable cannot be relaxed, without permitting K to become unbounded as an operator on L^2 or having Kf fail to exist pointwise, even for some continuous functions f, as M. Weiss and A. Zygmund have shown recently.

An elaboration of the preceding technique leads to results on the boundedness of the operators $\overline{K}f = \sup_\epsilon |K_\epsilon f|$ in various function spaces. The operator $\overline{K}$ is bounded with respect to a norm N if k satisfies the same conditions as above and $\overline{H}(\nu)$, $\overline{H}(\nu)f = \sup_\epsilon |H_\epsilon(\nu)f|$, is bounded with respect to N. This is the case, for example, if N is the norm of L^p, $1 < p < \infty$. Since for compactly supported differentiable f, $K_\epsilon f$ converges pointwise everywhere as ϵ tends to zero, and since every f in L^p can be written as $f = f_1 + f_2$ with f_1 differentiable and compactly supported and f_2 with arbitrarily small norm, it follows from the boundedness of $\overline{K}$ in L^p that $\lim \sup_{\epsilon \to 0} K_\epsilon f - \lim \inf_{\epsilon \to 0} K_\epsilon f$ has arbitrarily small norm in L^p. Thus Kf converges almost everywhere as ϵ tends to zero.

Later on we shall consider singular integral operators with variable kernels, that is operators of the form

$$(7) \qquad Kf = \lim_{\epsilon \to 0} \int_{|x-y|>\epsilon} k(x, x - y) f(y) \, dy,$$

where $k(x, z)$ is, for each x, a positively homogeneous function of degree $-n$ in z with mean value zero on $|z| = 1$. The same preceding technique can be used to study the existence and continuity of these operators. Let us mention just one result. If $1 < q < \infty$, and

$$\int_{|z|=1} | K(x, z) |^q \, d\sigma,$$

where $d\sigma$ denotes the surface area of the sphere $|z|=1$, is finite and bounded as a function of x, and f is in L^p, $q/(q-1)\leqq p<\infty$, then Kf exists pointwise almost everywhere and as a limit in L^p and K operates continuously on L^p.

Let us discuss now the Hilbert transform. It is clear by now that there is a close connection between the properties of the Hilbert transform and those of functions analytic in the upper half plane. A classical result is the theorem of M. Riesz which asserts that H is continuous in L^p, $1<p<\infty$. There are several proofs of this. For the sake of completeness of this exposition I will present one which I consider particularly simple. Let $w=u+iv$ be a complex number such that $u>0$. Then if $1<p<3$ there exist two constants c_1 and c_2 such that

$$|v|^p \leqq c_1u^p - c_2R(w^p), \qquad c_1 > 0, \qquad c_2 > 0.$$

This is readily verified by observing that for $|\arg w|$ close to $\pi/2$ we have $|v|^p\leqq -c_2R(w^p)$, and for $|\arg w|<(\pi/2)-\epsilon$, $|v|^p+c_2R(w^p)<c_1u^p$. Let now $u(t)$ be a nonnegative function in L^p of the real line and let $f(z)=f(x+iy)$ be defined by

$$f(z) = u(x, y) + iv(x, y) = \frac{1}{\pi i}\int_{-\infty}^{+\infty} \frac{u(t)}{t-z}\,dt, \qquad y > 0. \tag{8}$$

Then $u(x, y)>0$ and the preceding inequality applies to u, v and $w=f$. Integrating with respect to x we obtain

$$\int_{-\infty}^{+\infty} |v(x, y)|^p\,dx \leqq c_1\int_{-\infty}^{+\infty} u(x, y)^p\,dx - c_2\int_{-\infty}^{+\infty} R(f(x+iy)^p)\,dx.$$

But the last term vanishes, and since $u(x, y)$ is obtained from $u(t)$ by convolving this with the Poisson kernel $P(x, y)$ which is integrable with respect to x uniformly in y, Young's theorem on convolutions gives

$$\int_{-\infty}^{+\infty} |v(x, y)|^p\,dx \leqq c_1\int_{-\infty}^{+\infty} u(x, y)^p\,dx \leqq c\int_{-\infty}^{+\infty} u(t)^p\,dt.$$

Finally, comparing $v(x, y)$ with H_yu, one finds that $v(x, y)-H_yu$ is obtained from $u(t)$ by convolving this with a function which is integrable uniformly in y. Applying again Young's theorem we find that

$$\int_{-\infty}^{+\infty} |v(x, y) - H_yu|^p\,dx \leqq c\int_{-\infty}^{+\infty} u(t)^p\,dt$$

and this combined with the preceding inequality gives

$$\int_{-\infty}^{+\infty} | H_y u |^p \, dx \leqq c \int_{-\infty}^{+\infty} u(t)^p \, dt$$

with c independent of y. From this we obtain the same result for $H_\epsilon(\nu)$ by merely integrating. This same inequality for the remaining values of p is obtained by the well-known duality argument.

More precise information about the behaviour of the Hilbert transform is afforded by the following remarkable result of E. Stein and G. Weiss. If u is the characteristic function of a set of finite measure of the real line, then the distribution function of Hu depends only on the measure of the set. The following is a noncomputational proof. Assume for simplicity that the set E in question is a finite union of finite intervals. Let $f(z)$ be as in (8) where $u(t)$ is the characteristic function of E. Then $f(z)$ is analytic in the half plane $P=\{y\geqq 0\}$ except at the endpoints of the intervals of E and $\lim_{y\to\infty} yf(iy)=|E|/\pi$. Furthermore, $f(x)$ has real part equal to either 0 or 1, and $w=f(z)$ maps P into the strip $S=\{0\leqq R(w)\leqq 1\}$ and the boundary ∂P of P into the boundary ∂S of S. Let now D_s be the subset of ∂S defined by $|I(w)|>s>0$. Then $f^{-1}(D_s)$ is precisely the set of points where $|Hu|>s$. Let now $0\leqq h(w)\leqq 1$ be a harmonic function in S such that $h=1$ on D_s and $h=0$ on the remainder of ∂S. Then $h_1(z)=h[f(z)]$ is harmonic in P and $h_1=1$ on $f^{-1}(D_s)$ and $h=0$ on the remainder of ∂P. Thus h_1 is the Poisson integral of the characteristic function of $f^{-1}(D_s)$ and the measure of this set is given by $\pi\ \lim_{y\to\infty} yh_1(iy)$. But $\pi\ \lim_{y\to\infty} yh_1(iy)=\pi\ \lim_{y\to\infty} yh[f(iy)]$ and since $f(iy)\sim|E|/(\pi y)$ as $y\to\infty$, it follows that $\lambda(s)=|\{|Hu|>s\}|=|f^{-1}(D_s)|=|E|h'(0)$, which depends merely on s and the measure of E. The function $\lambda(s)$ can be calculated explicitly by taking E to be an interval and one finds that $\lambda(s)=2|E|$ $(\sinh s)^{-1}$. By integration one finds the same expression for the distribution function of $H(\nu)u$, where u is the characteristic function of a set E in R^n. This result can be formulated in terms of the nonincreasing rearrangement $f^*(t)$ of the function $|f|=|H(\nu)u|$, which is the function inverse to its distribution function, by

$$f^*(t) = \sinh^{-1}(2\,|\,E\,|\,/t) = 2= \int_0^\infty (t^2+4s^2)^{-1/2} u^*(s)\, ds$$

or in terms of the function $f^{**}(t)$, which is defined by

$$f^{**}(t) = \frac{1}{t}\int_0^t f^*(s)\, ds = \sup_{|D|=t} \frac{1}{t}\int_D |f(x)|\, dx,$$

$$f^{**}(t) = \frac{2}{t}\int_0^\infty \sinh^{-1}(t/2s) u^*(s)\, ds$$

Since f^{**} depends subadditively on f, as is readily seen from the second definition of f^{**} above, by integration we obtain

$$(H(\nu)f)^{**} = \frac{2}{t}\int_0^\infty \sinh^{-1}(t/2s)f^*(s)\,ds.$$

Thus if K is a singular integral operator in R^n with odd kernel, this combined with [6] gives

$$(Kf)^{**} \leqq \frac{\pi}{t}\int |k(\nu)|\,d\sigma \int_0^\infty \sinh^{-1}(t/2)f^*(s)\,ds.$$

This inequality gives accurate information about the distribution of values of Kf and can be used to establish the boundedness of K with respect to various norms.

One more important application of the complex method to the study of singular integral operators is the following. Let $a(x)$ be a function on the real line with a bounded derivative and let

$$g(x) = \lim_{\epsilon\to 0}\int_{|x-t|>\epsilon} \frac{a(x)-a(t)}{(x-t)^2} f(t)\,dt.$$

Then, if f belongs to L^p, $1<p<\infty$, the integral converges in the mean of order p as ϵ tends to zero, and the limit function $g(x)$ satisfies the inequality $\|g\|_p \leqq c_p \sup|a'(x)|\,\|f\|_p$. This result contains the classical result of M. Riesz to which it reduces in the case $a(x)=x$. Its proof is too complicated to be outlined here but we will discuss briefly some of its consequences. Let $k(x)$ be defined in R^n-0 and be homogeneous of degree $-n-1$. Suppose furthermore that $|k|\ \log^+|k|$ is integrable on the sphere $|x|=1$. Then if $a(x)$ is a function in R^n with bounded derivatives and $f(x)$ is a function in L^p, $1<p<\infty$, the integral

$$\lim_{\epsilon\to 0}\int_{|x-y|>\epsilon} [a(x)-a(y)]k(x-y)f(y)\,dy$$

exists as a limit in the mean of order p and represents a bounded operator in L^p. This is the n-dimensional version of the preceding result, to which it can be reduced by a method similar to the one discussed at the beginning of this paragraph. Another consequence of general relevance to the theory is this. Let K be an operator as in (3) whose kernel $k(x)$ has first order partial derivatives belonging to $L\log^+ L$ on the unit sphere $|x|=1$, and let $Af=a(x)f(x)$ where $a(x)$ is a function with bounded first order derivatives. Then the operators

$$\frac{\partial}{\partial x_i}(AK - KA), \qquad (AK - KA)\frac{\partial}{\partial x_i} \tag{9}$$

which are well defined for functions in C_0^∞, are bounded with respect to the norm of L^p, $1 < p < \infty$. This much for the complex method.

The direct method for the study of the singular integral operators is based on properties of these which are somewhat different from the ones we have used so far. The kernels are no longer required to be homogeneous functions but must be assumed to be somewhat smooth. This leads to results that are not only qualitatively but also quantitatively different. Essentially the method consists in obtaining estimates for the measures $|E_\lambda|$ of the sets E_λ where $|Kf| > \lambda > 0$, f being, say, a function in C_0^∞. The precise basic estimates are the following

$$\text{(i)} \quad |E_\lambda| \leqq c\lambda^{-1}\|f\|_1, \qquad \text{(ii)} \quad |E_\lambda| \leqq c\lambda^{-2}\|f\|_2^2, \tag{10}$$

where $\|f\|_1$ and $\|f\|_2$ denote the norms of f in L^1 and L^2 respectively. As is shown by the classical interpolation theorem of Marcinkiewicz and the standard duality argument, this implies that K is continuous with respect to a large class of norms, including among others, the norms of the spaces L^p, $1 < p < \infty$. The above estimates are consequences of two properties of the operator K. The first one is that K is bounded with respect to the norm of L^2, which, as is readily seen, implies (ii). The other is this. Let S be any sphere in R^n and $\bar{S}$ the sphere with the same center as S and twice its radius. Then there is a constant c such that

$$\int_{R^n - \bar{S}} |Kf|\, dx \leqq c\|f\|_1 \tag{11}$$

for every function f supported in S and with vanishing integral, c being independent of S and f. This will hold if, for example, the kernel k of K satisfies the following inequality

$$\int_{|x| > 2|y|} |k(x - y) - k(x)|\, dx \leqq c \tag{12}$$

with c independent of y.

Since the first of the postulated properties of K clearly implies (ii), one only has to show (i). The argument that follows is admittedly sketchy but it contains all essential steps.

Let f be nonnegative and integrable and let λ be a given positive number. Then (and this is the extension of the lemma of F. Riesz referred to at the beginning of the paragraph) there exists a sequence of disjoint cubes Q_j in R^n such that f has mean value between λ and

$2^n\lambda$ on each of the Q_j and $f \leqq \lambda$ outside $\bigcup Q_j$. This is shown by first partitioning R^n by means of nonoverlapping congruent cubes so large that f has mean value less than λ on each of them; then refining the partition by successive subdivision of the cubes into 2^n congruent parts and sorting out at each step those cubes where f has mean value larger than λ. It is easy to see that the mean value of f on the cubes that have been set aside cannot exceed $2^n\lambda$, and the theorem on differentiation of indefinite integrals shows that $f \leqq \lambda$ almost everywhere in the remainder of R^n. Set now $f = g + h$, where g coincides with f outside $\bigcup Q_j$ and with the mean value of f on Q_j in each Q_j. Then since $g \leqq 2^n\lambda$ we have

$$|\{|Kg| > \lambda\}| \leqq \lambda^{-2}\|Kg\|_2^2 \leqq c\lambda^{-2}\|g\|_2^2 \leqq c\lambda^{-2}2^n\lambda\|g\|_1 \leqq c2^n\lambda^{-1}\|g\|_1.$$

Now for h we have $h = \sum h_j$ where h_j equals f minus its mean value in Q_j and $h_j = 0$ elsewhere. Thus h_j is supported by Q_j and its integral vanishes.

Let now S_j be the sphere concentric with Q_j and with radius equal to the diameter of Q_j and let D be the complement of $\bigcup S_j$. Then according to (11) we have

$$\int_D |Kh|\, dx \leqq \sum \int_D |Kh_j|\, dx \leqq c \sum \|h_j\|_1 \leqq c\|f\|_1,$$

and consequently

$$|\{|Kh| > \lambda\}| \leqq \sum |S_j| + c\lambda^{-1}\|f\|_1 \leqq c\sum |Q_j| + c\lambda^{-1}\|f\|_1.$$

But since

$$|Q_j|^{-1}\int_{Q_j} f\, dx \geqq \lambda,$$

it follows that

$$\sum |Q_j| \leqq \lambda^{-1}\sum \int_{Q_j} f\, dx \leqq \lambda^{-1}\|f\|_1$$

which combined with the inequality above gives

$$|\{|Kh| > \lambda\}| \leqq c\lambda^{-1}\|f\|_1.$$

Consequently

$$\begin{aligned} |E_{2\lambda}| &= |\{|Kf| > 2\lambda\}| \\ &\leqq |\{|Kg| > \lambda\}| + |\{|Kh| > \lambda\}| \leqq c\lambda^{-1}\|f\|_1 \end{aligned}$$

which is the desired result.

The condition that K be bounded with respect to the norm of L^2 is not very suitable for many applications and for this reason it is desirable to have conditions on the kernel k to replace it. For example the following inequalities

$$(13)\qquad \left|\int_{s<|x|<t} k(x)\,dx\right| \leqq c, \qquad \int_{|x|<t} |x|\,|k(x)|\,dx \leqq ct$$

for all s and t, $0<s<t$, with c independent of s and t, and (12) imply that K is bounded with respect to the norm of L^2.

An interesting generalization of the preceding results is obtained by assuming that the function f in (3) has values in a Banach space B_1 and that the values of k are bounded operators on B_1 with values in a second Banach space B_2. Then the foregoing conclusions are still valid with absolute values replaced by the corresponding norms. If B_1 and B_2 are Hilbert spaces, even the conditions (13) apply. The relevance of this extension stems from the fact that it yields a number of classical inequalities and their generalizations. We shall illustrate this with just one example. Let $P(x, y)$ be the Poisson kernel for the upper half plane and for each x let $k(x)$ be the operator mapping the complex number a into the function $ay^{1/2}\partial P(x, y)/\partial y$ of L^2 $(0<y<\infty)$. Then the integral

$$y^{1/2}\int_{-\infty}^{+\infty} \frac{\partial}{\partial y} P(x-t, y)f(t)\,dt$$

can be regarded as an operator mapping numerical functions $f(t)$ on the real line into functions on the real line with values in L^2 $(0<y<\infty)$. This operator satisfies conditions (12) and (13). Let $g(x, y)$ be the value of the integral above and $g(x)$ the norm of $g(x, y)$ as an element of L^2 $(0<y<\infty)$. Then, if $F(x, y)$ denotes the Poisson integral of $f(t)$ we have

$$g(x) = \left[\int_0^\infty y\left[\frac{\partial}{\partial y}F(x, y)\right]^2 dy\right]^{1/2}$$

that is, $g(x)$ is precisely the Littlewood-Paley g-function associated with $f(t)$, and the results above yield the inequality $\|g\|_p \leqq c_p\|f\|_p$, $1<p<\infty$, between the L^p-norms of f and g.

Further results on the continuity of singular integral operators can be obtained from the fact that they commute with translations. Let N be a norm defined on a linear class M of locally integrable functions containing C_0^∞. Let h be a function in C_0^∞ with the property that the convolution $g*h$ of g and h is in M whenever g is, and let $N_h(g)$

$=N(g*h)$. If K is now a singular integral operator which is bounded with respect to N, since $h*Kg=K(h*g)$ for every g in C_0^∞, we have $N_h(Kg)=N(h*Kg)=N(K(h*g))\leqq cN(h*g)=cN_h(g)$. Thus K is also bounded with respect to N_h and its norm with respect to N_h does not exceed its norm with respect to N. Moreover, by a passage to the limit, it follows that K is bounded with respect to any norm which is a limit of such N_h. In this fashion it is possible to show that singular integral operators are bounded with respect to the norms of various spaces of differentiable functions and distributions. An example of such spaces is the space L_k^p of functions in L^p with derivatives up to order k in L^p, $1<p<\infty$, and the space L_{-k}^q of distributions which are continuous on L_k^p.

To conclude our discussion of the continuity of singular integral operators, I would like to mention the fact that many local properties of functions are stable under the action of these operators. For example, let T_u^p, $1<p<\infty$, $u>0$ and not equal to an integer, be the class of functions $f(x)$ in L^p of R^n having a Taylor expansion at the origin of the form

$$f(x) = P(x) + R(x)$$

where $P(x)$ is a polynomial of degree less than u and $R(x)$ is such that

$$\left[\frac{1}{\rho^n}\int_{|x|<\rho} | R(x) |^p \, dx\right]^{1/p} \leqq c\rho^u$$

for all $\rho>0$. Then if the kernel k of K is sufficiently smooth, K maps T_u^p continuously into itself (T_u^p has a natural norm). An interesting consequence of this is the following; if C is a closed set in R^n and $f(x)\in L^p$ has the property that

$$| f(x) - f(y) | \leqq c | x - y |^u, \qquad 0 < u < 1$$

for x in C, then the same is true of Kf for x and y in C. This generalizes a classical result of Privalov about the conjugate function.

3. **Singular integrals and multipliers of Fourier transforms.** Let $b(x)$ be a bounded function in R^n and let B be the operator defined by $(Bf)^\wedge = b\hat{f}$, where $\hat{f}$ is the Fourier transform of f. The function b is said to be a multiplier for the linear class M of functions f, if $f\in M$ implies $Bf\in M$. Evidently, if the singular integral operator K maps the class M into itself then $\sigma(K)$ is a multiplier for the class M. Thus results on the continuity of singular integral operators can be restated as theorems on multipliers. In this framework it is possible to obtain partial generalizations of the results in the previous section. In this

connection let us mention a theorem of Hörmander which in a simplified version can be stated as follows: let $b(x)$ be bounded and let the derivatives $D_k b$ of b of order k be such that $|D_k b(x)| \leqq c|x|^{-k}$ for all k less than or equal to $1+n/2$, then b is a multiplier for L^p, $1<p<\infty$. A similar result for operator valued multipliers has been obtained by S. Vagi.

4. Singular integral operators and ergodic theory. The direct method for the study of the continuity of singular integral operators has a number of features in common with those employed in ergodic theory and in the theory of differentiation of indefinite integrals. In fact, it is possible to obtain some of the basic results in the corresponding cases as special cases of a general theorem due to M. Cotlar. Let $\mathfrak{M}$ be a totally σ-finite measure space and $G=\{T_y\}$, $y\in R^n$, an n-parameter group of measure preserving transformations of $\mathfrak{M}$. Let $h(y)$ be an integrable function in R^n such that

$$h(y) = 0 \qquad \text{for } |y| \geqq 2, \qquad \int h(y)dy = 0,$$

$$\int |h(y-z) - h(y)|\, dy \leqq c|z|$$

with c independent of z. Then, if $f(x)$ is a function in $L^p(\mathfrak{M})$, $1<p<\infty$,

$$\tilde{f}_{N,M}(x) = \int \sum_{-N}^{M} 2^{nj}h(2^j y)f(T_y x)\, dy$$

converges in the mean of order p and pointwise almost everywhere to a limit function $\tilde{f}$ in $L^p(\mathfrak{M})$ as N or M tend to infinity. Furthermore, if $\bar{f}=\sup|\tilde{f}_{N,M}|$, then $\bar{f}$ is in $L^p(\mathfrak{M})$ and $\|\bar{f}\|_p \leqq c\|f\|_p$.

If we set $h(y)=1-2^n$ for $|y|\leqq 1$ and $h(y)=1$ for $1<|y|<2$ we obtain

$$\tilde{f}_{N,0}(x) = 2^{-nN}\int_{|y|<2^{N+1}} f(T_y x)\, dy - 2^n \int_{|y|<1} f(T_y x)\, dy,$$

$$\tilde{f}_{0,M}(x) = -2^{n(M+1)}\int_{|y|<2^{-M}} f(T_y x)\, dy + \int_{|y|<2} f(T_y x)\, dy,$$

and the preceding statement applied to $\tilde{f}_{N,0}$ and $\tilde{f}_{0,M}$ yields the ergodic maximal ergodic and differentiation theorems for functions in $L^p(\mathfrak{M})$, $1<p<\infty$, as can be seen without difficulty. If on the other hand $\mathfrak{M}=R^n$ and T_y is translation by y, setting $h(y)=k(y)$ if $1<|y|<2$ and $h(y)=0$ otherwise, where k is as in (3), we obtain

$$\tilde{f}_{NM}(x) = \int_{2^{-M}<|y|2^{N+1}} k(y)f(x-y)\,dy$$

which is essentially the integral in (3).

5. **Parabolic singular integral operators.** Divergent convolution integrals analogous to the ones we have been considering appear in a number of analytical problems. An interesting example is that of the study of the highest order derivatives of solutions of parabolic equations with constant coefficients. This leads to integrals of the following form

$$Kf = \lim_{\epsilon\to 0} K_\epsilon f, \qquad K_\epsilon f = \int_{|t-s|>\epsilon} k(x-y, t-s)f(y, s)\,dx\,ds$$

where x and y are points in R^n and s and t range over the real line R, and $k(x, t)$ has the following homogeneity property

$$k(\lambda x, \lambda^m t) = \lambda^{-n-m}k(x, t) \qquad \lambda > 0$$

m being a positive integer. Under some additional assumptions on the kernel k many of the results valid for operators as in (3) remain valid in this more general situation. For example, if $f\in L^p(R^{n+1})$, $1<p<\infty$, then $K_\epsilon f$ converges almost everywhere and in the mean of order p to a limit. Furthermore, $\sup_\epsilon |K_\epsilon f|$ belongs to L^p and its norm does not exceed a constant depending on k times the L^p-norm of f.

Sufficient conditions on k for the validity of these results are the following:

$k(x, t) = 0$ for $t < 0$,

$$\int_{R^n} k(x, 1)(1 + |x|)\,dx < \infty, \qquad \int k(x, 1)\,dx = 0,$$

$$\int_{t>2|s|} |k(x-y, t-s) - k(x, t)|\,dx\,dt < c,$$

where c is independent of y and s. Under these assumptions it is not difficult to show that the Fourier transform of the function k_ϵ, $k_\epsilon(x, t) = k(x, t)$ if $t>\epsilon$ and $k_\epsilon = 0$ otherwise, is bounded in ϵ. Furthermore, a suitable modification of the direct method for the operators in (3), yields the inequalities (10) which are also valid in the present case. The argument used there was based on an extension of the lemma of F. Riesz, obtained by successive subdivisions of R^n by means of congruent nonoverlapping cubes. A careful examination of that argument shows that a similar procedure is applicable in the present situation. One merely has to use parallelopipeds instead of cubes, these parallelopipeds being cartesian products of cubes Q in the space R^n of the variable x and intervals I in the space of the variable t, such that $|Q|^{m/n} = |I|$.

6. Singular integral operators with variable kernels. As we have seen, differential operators with constant coefficients and of homogeneous order can be represented as the product of a singular integral operator and a power of the operator Λ. An analogous representation can be obtained for operators with variable coefficients if the singular integral operators are suitably generalized. For this purpose it is enough to consider operators as in (7), where again, $k(x, z)$ is positively homogeneous of degree $-n$ (here, as before, n is the number of variables of the functions on which the operators act) and has mean value zero on $|z| = 1$. However, for the purpose of applications to partial differential equations, it is sufficient to consider kernels which are smooth. We shall say that the operator K is of class m if $k(x, z)$ is infinitely differentiable with respect to z and m times differentiable with respect to x in $|z| > 0$ and

$$\left|\left(\frac{\partial}{\partial x}\right)^{\alpha}\left(\frac{\partial}{\partial z}\right)^{\beta} k(x, z)\right| \leqq c_{\beta} \tag{14}$$

on $|z| = 1$, where $(\partial/\partial x)^{\alpha}$ represents differentiation with respect to the coordinates of x of zero or positive order not exceeding m, and $(\partial/\partial z)^{\beta}$ represents differentiation with respect to coordinates of z of any order. We shall henceforth consider only operators of this type.

In studying continuity properties of these operators one may use the method of reduction to the Hilbert transform described in §2. However, given our present assumptions on the kernel $k(x, z)$, it is perhaps more convenient to proceed as follows. Let $Y_j(z)$ be a sequence of positively homogeneous functions of degree zero coinciding with a complete set of normalized spherical harmonics on $|z| = 1$, arranged in order of nondecreasing degree. Then $k(x, z)$ can be expanded in series

$$k(x, z) = \sum_{1}^{\infty} a_j(x) Y_j(z) \, |z|^{-n},$$

where, given our assumptions on $k(x, z)$, the functions $a_j(x)$ converge rapidly to zero, that is $a_j(x) = O(j^{-r})$ uniformly in x for all $r > 0$. Correspondingly we may expand the operator K in series

$$K = \sum_{1}^{\infty} A_j R_j, \tag{15}$$

where $A_j f = a_j(x)f(x)$ and R_j is the translation invariant singular integral operator with kernel $Y_j(x-y)|x-y|^{-n}$. Given the continuity properties of the R_j and the rapid decrease of the a_j, it is easy to deduce continuity properties of K from the preceding expansion. For example, K is continuous with respect to the norm of $L_k^p(R^n)$, $1<p<\infty$, $-m\leqq k\leqq m$, as follows readily from the fact that the R_j are uniformly bounded on $L_k^p(R^n)$ and the a_j and their derivatives of orders less than or equal to m converge rapidly to zero.

As in the case of differential operators with constant coefficients, in order to obtain a representation of differential operators with variable coefficients it will be necessary to consider slightly more general operators, namely those of the form

$$Hf = a(x)f(x) + Kf,$$

where K is of class m and $a(x)$ is bounded and has bounded derivatives up to order m. These we shall also call singular integral operators of class m. With such operators we associate their symbols $\sigma(H)$, which are functions of two arguments defined by

$$\sigma(H) = a(x) + k(x, z)^\wedge$$

where $k(x, z)^\wedge$ is the Fourier transform of $k(x, z)$ with respect to z. It is not difficult to show that if H is an operator of class m, then $\sigma(H)$ is homogeneous of degree zero with respect to z, and is m times differentiable with respect to x and infinitely differentiable with respect to z in $|z|>0$. Furthermore, $\sigma(H)$ satisfies inequalities similar to those in (14). Conversely, every function of x and z with these properties is the symbol of a unique operator of class m.

Singular integral operators with variable kernels do not form a class closed under composition. However, there is an approximate functional calculus for such operators which is an adequate substitute for a true functional calculus and which has the advantage of having a much simpler algebraic structure than might be expected of the latter. To describe this approximate functional calculus we have to introduce some additional notions. We will say that an operator S is smoothing of class m, $m\geqq 1$, if it is defined in

$$\sum_{1<p<\infty} L_{-m}^p(R^n)$$

and maps L_r^p continuously into L_{r+1}^p for all p, $1<p<\infty$, and $-m\leqq r \leqq m-1$. Given two singular integral operators H_1 and H_2 of class m the functions $\sigma(H_1)\sigma(H_2)$ and $\bar{\sigma}(H_1)$ (i.e., the complex conjugate of

$\sigma(H_1)$) are themselves symbols of operators of class m. Thus we may define the pseudoproduct $H_1 \circ H_2$ of the operators H_1 and H_2 and the pseudoadjoint $H^{\#}$ of the operator H by

$$(16) \qquad \sigma(H_1 \circ H_2) = \sigma(H_1)\sigma(H_2), \qquad \sigma(H^{\#}) = \bar{\sigma}(H_1).$$

With this law of composition the class of singular integral operators of class m becomes an algebra with a conjugation which is isomorphic to the algebra of all functions $f(x, z)$ which are positively homogeneous of degree zero with respect to z, are m times differentiable with respect to x, infinitely differentiable with respect to z, and satisfy the inequalities (14). The interest of this rather obvious point of view lies in the less obvious fact that in some sense the pseudoproduct and the pseudoadjoints of operators are not very different from their true product and adjoints. Specifically, $H_1 \circ H_2 - H_1H_2$ and $H^{\#} - H^*$, where H^* denotes the adjoint or transposed conjugate of H, are smoothing operators of class m. Before we sketch the proof of this, let us state this result in a more algebraic language. First let us observe that since a singular integral operator H of class m maps L^p_r continuously into itself for all p, $1 < p < \infty$, and $-m \leqq r \leqq m$, if S is a smoothing operator of class m then HS and SH are also smoothing of class m. Furthermore, if S is smoothing of class m so is S^*. Thus the class $\mathcal{S}_m$, $m \geqq 1$, of operators of the form $H+S$ where H is a singular integral operator of class m and S is smoothing of class m, is closed under composition and with respect to the operation of taking adjoints, and the smoothing operators form a two-sided self-adjoint ideal $\mathfrak{J}_m$ in this algebra. Reducing the algebra modulo $\mathfrak{J}_m$ one obtains the algebra of singular integral operators of class m with the pseudoproduct as multiplication. This is a consequence of the fact that a singular integral operator is smoothing only if it vanishes. Two important properties of $\mathcal{S}_m$ which play an important role in applications are the following: if H is a singular integral operator in $\mathcal{S}_m$ then $(\partial/\partial x_j)H - H(-\partial/\partial x_j)$ is a singular integral operator in $\mathcal{S}_{m-1}$ and $\Lambda H - H\Lambda$ is an operator in $\mathcal{S}_{m-1}$. If in addition $\sigma(H)$ does not vanish then there exists an S in $\mathfrak{J}_m$ such that $H+S$ has a two-sided inverse in $\mathcal{S}_m$.

Let us now describe briefly how one shows that $H_1 \circ H_2 - H_1H_2$ is smoothing, a similar argument being applicable to $H^{\#} - H^*$. Let us assume first that $H_1 = A_1K_1$ and $H_2 = A_2K_2$, where K_1 and K_2 are translation invariant singular integral operators and A_1 and A_2 are multiplication by the functions $a_1(x)$ and $a_2(x)$ respectively. In this case one can readily see that

$$H_1H_2 = A_1K_1A_2K_2, \qquad H_1 \circ H_2 = A_1A_2K_1K_2,$$

and consequently,

$$H_1 \circ H_2 - H_1H_2 = A_1(A_2K_1 - K_1A_2)K_2.$$

Now the operators K_1 and K_2 map L^p_r continuously into itself for all r, and the same is true for A_1 and A_2 and $-m \leqq r \leqq m$ provided that the functions $a_1(x)$ and $a_2(x)$ are bounded and have bounded derivatives up to order m. Thus if $A_2K_1 - K_1A_2$ is smoothing of class m the same holds for $H_1 \circ H_2 - H_1H_2$. Now, there is the following criterion for an operator B mapping L^p_r continuously into itself for $-m \leqq r \leqq m$, to be smoothing of class m. For $m=1$ it is necessary and sufficient that the operators $(\partial/\partial x_j)B$ and $B(\partial/\partial x_j)$ map L^p continuously into itself. For $m>1$ it is sufficient that the operators $(\partial/\partial x_j)B - B(\partial/\partial x_j)$ be smoothing of class $m-1$. This criterion is readily verified using the fact that every element g of L^p_r can be respresented as

$$g = g_0 + \sum_1^n \frac{\partial}{\partial x_j} g_j$$

with g_j, $0 \leqq j \leqq n$, in L^p_{r+1}. Applying this criterion to $A_2K_1 - K_1A_2$, the case $m=1$ leads to operators like those in (9), which, as was mentioned there, satisfy the conditions above. The case $m>1$ then follows by induction. Once the desired result has been established for H_1 and H_2 as above, the general case follows by expansion in series like in (15).

Let us turn now to the representation of differential operators. Suppose that D is a monomial differential operator

$$D = a(x)\left(\frac{\partial}{\partial x_1}\right)^{\alpha_1}\left(\frac{\partial}{\partial x_2}\right)^{\alpha_2}\cdots\left(\frac{\partial}{\partial x_n}\right)^{\alpha_n},$$

where $a(x)$ is bounded and has bounded derivatives up to order m. Then

$$D = AK\Lambda^r = H\Lambda^r, \qquad r = \alpha_1 + \alpha_2 + \cdots + \alpha_n$$

where A is the operator multiplication by $a(x)$, K is a translation invariant singular integral operator and H is a singular integral operator in $\mathcal{S}_m$ with

$$\sigma(K)(x) = (2\pi i)^r x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n} \left| x \right|^{-r}, \qquad \sigma(H) = a(x)\sigma(K)(z).$$

If now D is a general differential operator of order r with coefficients as above we have

$$D = H\Lambda^r + \sum_1^r H_j\Lambda^{r-j}$$

where the H are singular integral operators in $\mathcal{S}_m$, with

$$\sigma(H)(x, z) = (2\pi i)^r P(x, z) \mid z \mid^{-r}. \tag{17}$$

$P(x, z)$ being the characteristic polynomial of D. A shorter representation of D can be obtained if one observes that according to the multiplier theorem of Hörmander, $(I+\Lambda)^{-j}$ is a smoothing operator of class m for all m and $j>0$. Thus setting $\Lambda^j=[(I+\Lambda)-I]^j$ in the expression for D above, expanding and factoring out $(I+\Lambda)^r$ on the right. Thus one obtains

$$D = (H + S)(I + \Lambda)^r \tag{18}$$

where S is a smoothing operator of class m and H is a singular integral operator of class m whose symbol is given by (17).

7. **Singular integral operators on manifolds.** The theory of singular integral operators with variable kernel can be extended to manifolds. For the sake of simplicity we shall restrict our discussion to the compact infinitely differentiable case. Let us begin with some remarks about the algebra of operators $\mathcal{S}_m$ introduced in the preceding section. We may define the symbol $\sigma(A)$ for every operator in $\mathcal{S}_m$ by $\sigma(A)=\sigma(H)$ where $A=H+S$, H being a singular integral operator and S being an element of $\mathfrak{I}_m$. Then $\sigma(A)$ is well defined and gives a homomorphism of $\mathcal{S}_m$ onto the algebra of symbol functions. The kernel of this homomorphism consists precisely of the smoothing operators. If we regard R^n as an n-dimensional vector space T, we can think of the symbol $\sigma(A)(x, z)$ of an operator A, which for each x is the Fourier transform of a distribution on R^n or T, as a function defined on the cartesian product of R^n and the dual T^* of T. Now, this cartesian product is canonically isomorphic with the cotangent bundle $T^*(R^n)$ of R^n. Thus the symbol can be regarded as a function on $T^*(R^n)$. The symbol functions are characterized by their differentiability properties as described in the preceding paragraph and the fact that their restrictions to fibres are positively homogeneous functions of degree zero. Given a compact infinitely differentiable manifold $\mathfrak{M}$ we can similarly introduce the symbols of class m on $\mathfrak{M}$ as functions on cotangent bundle $T^*(\mathfrak{M})$ of $\mathfrak{M}$ which are positively homogeneous of degree zero on each fibre and have the corresponding differentiability properties. We can also define the distribution spaces $L_r^p(\mathfrak{M})$, $1<p<\infty$, $-\infty<r<\infty$, as the class of distributions on $\mathfrak{M}$ which coincide on coordinate neighborhoods with distributions in $L_r^p(R^n)$. We shall say that an operator defined $\sum_{p>0} L_{-m}^p(\mathfrak{M})$ is smoothing of class m if it maps $L_r^p(\mathfrak{M})$ continuously into $L_{r+1}^p(\mathfrak{M})$ for $-m\leqq r\leqq m-1$ and $1<p<\infty$. We will denote this class of operators $\mathfrak{I}_m(\mathfrak{M})$.

We can now define singular integral operators on $\mathfrak{M}$. This is done essentially by transplanting operators from Euclidean space. Specifically, an operator A defined on $\sum_{p>1} L^p_{-m}(\mathfrak{M})$ will be called a singular integral operator of class m, i.e., an operator in $\mathcal{S}_m(\mathfrak{M})$, if it has the following properties:

(i) if ϕ_1 and ϕ_2 are infinitely differentiable functions on $\mathfrak{M}$ with disjoint supports then N_1AN_2, where N_1 and N_2 are the operators multiplication by ϕ_1 and ϕ_2 respectively, is an operator in $\mathfrak{I}_m(\mathfrak{M})$;

(ii) there exists a symbol $\sigma(A)$ of class m on $\mathfrak{M}$, such that for every C^∞ diffeomorphism δ of an open subset W of R^n on an open subset V of $\mathfrak{M}$ and any two infinitely differentiable functions ϕ_1 and ϕ_2 with support in V; there exists an operator B in $\mathcal{S}_m$ with the property that $(N_1AN_2f)\circ\delta = B(f\circ\delta)$ for every C^∞ function f with support in V, and $[\bar{\phi}_1\sigma(A)\bar{\phi}_2]\circ\bar{\delta} = \sigma(B)$, where N_1 and N_2 are the operators multiplication by ϕ_1 and ϕ_2 respectively, $\bar{\phi}_j$ is the function ϕ_j lifted to $T^*(R^n)$ and $\bar{\delta}$ is the mapping of the portion of $T^*(R^n)$ above W into $T^*(\mathfrak{M})$ induced by δ.

It is not clear that there exist operators in $\mathcal{S}_m(\mathfrak{M})$ with nonvanishing symbols. One can show, however, that the operators in the $\mathcal{S}_m$ of the preceding section, which according to our present notation will henceforth be denoted by $\mathcal{S}_m(R^n)$, indeed satisfy the conditions (i) and (ii) above. For this it is necessary to show that, roughly speaking, substitution of variables in operators in $\mathcal{S}_m(R^n)$ leads to operators of the same kind. This requires calculations which are somewhat complicated, but the argument is otherwise standard except at one point. If $k(x, x-y)$ is the kernel of a singular integral operator in $\mathcal{S}_m(R^n)$ and $x = F(\bar{x})$ is a C^∞ substitution of variables one can show that

$$k[F(\bar{x}), F(\bar{x}) - F(\bar{y})] = k_1(\bar{x}, \bar{x}-\bar{y}) + k_2(\bar{x}, \bar{x}-\bar{y}) + R(\bar{x}, \bar{x}-\bar{y}),$$

where $k_1(\bar{x}, \bar{x}-\bar{y})$ and $k_2(\bar{x}, \bar{x}-\bar{y})\,|\bar{x}-\bar{y}|^{-1}$ are kernels of operators in $\mathcal{S}_m(R^n)$,[2] and where $R(\bar{x}, \bar{z})$ is m times differentiable with respect to $\bar{x}$ and infinitely differentiable with respect to $\bar{z}$. Furthermore R and its derivatives up to order m with respect to coordinates of $\bar{x}$ and order 1 with respect to coordinates of $\bar{z}$, are integrable with respect to $\bar{z}$ near $\bar{z}=0$, uniformly in $\bar{x}$. Thus, using the criterion described in the preceding section, one can show that the operator with kernel $R(\bar{x}, \bar{x}-\bar{y})$ is, at least locally, smoothing of class m. Similarly one can show that the operator with kernel $k_2(\bar{x}, \bar{x}-\bar{y})$ is locally smoothing of class m. Once this has been established one encounters no further difficulties in showing that operators in $\mathcal{S}_m(R^n)$ satisfy (ii). With this information about $\mathcal{S}_m(R^n)$ it is possible to give a standard method for constructing operators in $\mathcal{S}_m(\mathfrak{M})$ with prescribed symbol. For this

[2] For $k_2(\bar{x}, \bar{z})\,|\bar{z}|^{-1}$ this is true except for the vanishing of its mean value on $|\bar{z}| = 1$.

purpose we take any partition of unity on $\mathfrak{M}$ $1=\sum\phi_j^2$, such that ϕ_j is infinitely differentiable and has support in a coordinate neighborhood V_j, and let δ_j be diffeomorphisms of open subsets W_j of R^n onto V_j. Given a symbol function σ on $T^*(\mathfrak{M})$ we let B_j be singular integral operators in R^n such that $\sigma(B_j)=\sigma\circ\bar{\delta}_j$, where $\bar{\delta}_j$ is the mapping of the portion of $T^*(R^n)$ above W_j into $T^*(\mathfrak{M})$ induced by δ_j, and define the operators A_j by

$$(A_jf)\circ\delta_j = (\phi_j\circ\delta_j)B_j[(\phi_j\circ\delta_j)(f\circ\delta_j)]$$

in V_j and $A_jf=0$ outside V_j. Finally we set $A=\sum A_j$. Then A is an operator in $\mathcal{S}_m(\mathfrak{M})$ and $\sigma(A)=\sigma$. This can be shown without difficulty using partitions of unity and the fact that operations in $\mathcal{S}_m(R^n)$ satisfies conditions (i) and (ii).

Let us return now to the discussion of $\mathcal{S}_m(\mathfrak{M})$. By using again partitions of unity one can show that this class is closed under composition. Since $\sigma(A)$ is clearly multiplicative and linear, it gives a homomorphism of the algebra $\mathcal{S}_m(\mathfrak{M})$ onto the algebra of symbols of class m. The kernel of this homomorphism is precisely $\mathfrak{I}_m(\mathfrak{M})$, i.e., the class of smoothing operators of class m. A smoothing operator of class m can also be regarded as a mapping of $L_r^p(\mathfrak{M})$, $1<p<\infty$, $-m\leqq r\leqq m-1$, into itself, and as such, according to a well-known theorem of Rellich, it is completely continuous. Thus the kernel of the symbol homomorphism consists of completely continuous operators of this kind. Conversely, one can prove that if an operator in $\mathcal{S}_m(\mathfrak{M})$ is completely continuous as an operator in one of the spaces $L_r^p(\mathfrak{M})$, $-m\leqq r\leqq m-1$, then it belongs to $\mathfrak{I}_m(\mathfrak{M})$. The method for constructing operators in $\mathcal{S}_m(\mathfrak{M})$ with prescribed symbol actually yields a linear mapping of the space of symbols of class m into $\mathcal{S}_m(\mathfrak{M})$ which is a right inverse of the symbol homomorphism, and which we will henceforth denote by τ. This right inverse of the symbol homomorphism is of course not unique, for it depends on the choice of the partition $1=\sum\phi_j^2$ and the diffeomorphisms δ_j. Nevertheless it has some other important properties which we will describe briefly. First of all, it is continuous in the following sense. If a sequence of symbol functions σ_j and their derivatives of order up to m with respect to coordinates of $\mathfrak{M}$ and order $2n$ with respect to coordinates of the fibers of $T^*(\mathfrak{M})$ converge uniformly on compact subsets of $T^*(\mathfrak{M})-\mathfrak{M}$, then the operators $\tau(\sigma_j)$ converge with respect to their norms as operators in $L_r^p(\mathfrak{M})$, $1<p<\infty$, $-m\leqq r\leqq m$. This can be used to extend $\mathcal{S}_m(\mathfrak{M})$ to a larger algebra which we will denote by $\bar{\mathcal{S}}_m(\mathfrak{M})$, in such a way that every function which is the limit of a sequence of symbols which converges in the sense described above is the symbol of an operator in $\bar{\mathcal{S}}_m(\mathfrak{M})$. All properties of $\mathcal{S}_m(\mathfrak{M})$ discussed so far are shared by $\mathcal{S}_m(\mathfrak{M})$.

A second significant property of τ is this: operators in the range of τ nearly preserve the support of functions. Specifically, if a metric is given on $\mathfrak{M}$, and the functions ϕ_j used in the construction of τ have supports with diameters not exceeding ϵ, then for every A in the range of τ the support of Af is contained in an ϵ-neighborhood of the support of f. This is readily seen from the construction of τ. A last property of $\mathcal{S}_m(\mathfrak{M})$ which we would like to mention is that it is closed with respect to transposition relative to any C^∞ positive density on $\mathfrak{M}$. This is also readily verified from the definition of $\mathcal{S}_m(\mathfrak{M})$.

As in the case of Euclidean space, linear differential operators on a compact manifold $\mathfrak{M}$ can be represented in terms of operators in $\mathcal{S}_m(\mathfrak{M})$. Given a linear differential operator D of order r on $\mathfrak{M}$, there is associated with it its characteristic polynomial $\sigma(D)$, which is a function on $T^*(\mathfrak{M})$ coinciding on each fiber with a homogeneous polynomial of degree r. If the coefficients of D have bounded derivatives up to order m, then $\sigma(D)$ has the same differentiability properties as the symbols of class m. Now, one can construct an operator acting on functions on a compact manifold which is analogous to the operator Λ of the Euclidean case, and which we shall also denote by Λ. Its properties are the following: $(I+\Lambda)$ maps $L_r^p(\mathfrak{M})$ isomorphically onto L_{r-1}^p for $1<p<\infty$, and $-\infty<r<\infty$ (actually, Λ is also defined on the space of distributions on $\mathfrak{M}$); associated with Λ there is a function $\sigma(\Lambda)$ on $T^*(\mathfrak{M})$ which is positively homogeneous of degree 1 on the fibers and infinitely differentiable and positive on $T^*(\mathfrak{M})-\mathfrak{M}$ and such that for every linear differential operator D of order r with coefficients m times boundedly differentiable, there exists a unique operator A in $\mathcal{S}_m(\mathfrak{M})$ for which

$$D = A(I+\Lambda)^r, \qquad \sigma(D) = \sigma(A)\sigma(\Lambda)^r; \tag{19}$$

for every A in $\mathcal{S}_m(\mathfrak{M})$ the operator $A\Lambda-\Lambda A$ maps $L_r^p(\mathfrak{M})$ continuously into itself for $1<p<\infty$ and $-m+1\leqq r\leqq m-1$.

The construction of this operator Λ is relatively simple. Let us consider the functions ϕ_j and the diffeomorphisms δ_j and $\bar{\delta}_j$ that we employed in the construction of τ and let us denote by Λ_E the Euclidean Λ as defined in (4) and by $\sigma(\Lambda_E)$ the function $|x|$ on the cotangent space $T^*(R^n)$. Then if σ_j are functions on $T^*(\mathfrak{M})$ such that

$$\sigma_j \circ \bar{\delta}_j = \sigma(\Lambda_E)$$

on the portion of $T^*(R^n)$ above W_j, and Λ_j are the operators defined by

$$(\Lambda_j f) \circ \delta_j = (\phi_j \circ \delta_j)\Lambda_E[(\phi_j \circ \delta_j)(f \circ \delta_j)]$$

on V_j and $\Lambda_j f=0$ outside V_j, we define $\sigma(\Lambda)=\sum\phi_j\sigma_j\phi_j$ and $\Lambda=\sum\Lambda_j$. The properties of the operator Λ can be readily surmised from its definition and the corresponding properties of the Euclidean Λ, however their proof is somewhat technical and for this reason it will not be outlined here. One important property though, is an immediate consequence of the definition, namely that of near preservation of support. If a metric is given on $\mathfrak{M}$ and the supports of the ϕ_j have diameter not exceeding ϵ, the support of Λf is contained in an ϵ-neighborhood of the support of f.

8. **More refined calculi of singular integral operators.** As we shall see, in many applications of the theory of singular integral operators it is possible to neglect operators which are smoothing. In these situations one can restrict oneself to essentially manipulate symbols instead of operators and the theory becomes a flexible and powerful tool. However, the scope of this technique is limited by its crudeness and many problems require a more careful treatment. Thus there arises naturally the need to refine the theory. In a recent paper J. Kohn and L. Nirenberg have done this for operators acting on functions on Euclidean space (see also [46] to [50]). In this section I will describe the corresponding theory for manifolds. In many respects it resembles the one discussed in the preceding section. The algebra of symbols however, is much more complicated and, among other things, is noncommutative. No indication of proofs will be given.

Let us begin discussing the Euclidean case. We will denote by $\mathcal{S}^r_\infty(R^n)$, or by simply $\mathcal{S}^r(R^n)$, (we will restrict our discussion to the C^∞ case), the class of operators of rank r and class ∞, that is operators on $\sum L^p(R^n)$, $1<p<\infty$, of the form

$$A = H_0 + \sum_1^r H_j + S \tag{20}$$

where H_0 is a singular integral operator belonging to $\mathcal{S}_m(R^n)$ for all m, S is an operator mapping $L^p_m(R^n)$ continuously into $L^p_{m+r+1}(R^n)$ for $1<p<\infty$ and $-\infty<m<\infty$, and

$$H_j f = \int k_j(x, x-y)f(y)\,dy, \qquad j \geqq 1$$

where the Fourier transform $k_j(x, z)\hat{}$ of $k_j(x, z)$ with respect to z is bounded, has bounded derivatives of all orders and coincides for $|z| \geqq 1$ with a function $h_j(x, z)$ which is positively homogeneous of degree $-j$. Operators S with the property above we shall call smoothing of order $r+1$ and the class of such operators will be denoted by $\mathfrak{I}^{r+1}(R^n)$.

A first result in the theory is that $\mathcal{S}^r(R^n)$ is an algebra closed under transposition and $\mathfrak{I}^{r+1}(R^n)$ is a two-sided ideal in $\mathcal{S}^r(R^n)$. Furthermore $\mathcal{S}^r(R^n)$ is locally invariant under coordinate transformations. Specifically, given open subsets V and W of R^n, a C^∞ diffeomorphism δ of W onto V, two infinitely differentiable functions ϕ_1 and ϕ_2 with support in V and an operator A in $\mathcal{S}^r(R^n)$, there exists an operator B in $\mathcal{S}^r(R^n)$ such that

$$(N_1AN_2f)\circ\delta = B(f\circ\delta) \tag{21}$$

for every infinitely differentiable function f with support in V, where N_1 and N_2 denote the operators multiplication by ϕ_1 and ϕ_2 respectively. A second result is that operators in $\mathcal{S}^r(R^n)$ map $L^p_m(R^n)$ continuously into itself for $1<p<\infty$ and $m\geqq 0$, and are bounded with respect to the norm of $L^p_m(R^n)$ for $1<p<\infty$ and $m<0$.

To define the symbols of the operators in $\mathcal{S}^r(R^n)$ we have to introduce first some algebraic notions. Let X be a vector space over the reals and consider linear differential operators on X whose coefficients are positively homogeneous infinitely differentiable functions of nonpositive degree. We define the weight of a monomial operator of this kind to be its order minus the degree of its coefficient, and the weight of a multinomial operator as the minimum of the weights of its terms. These operators form an algebra $\mathfrak{A}(X)$ and the operators of weight larger than r form a two sided ideal $\mathfrak{F}^{r+1}(X)$ in $\mathfrak{A}(X)$. We define $\mathfrak{A}^r(X)$ to be the corresponding quotient algebra. In other words, $\mathfrak{A}^r(X)$ is the algebra of operators with coefficients of nonpositive degree and with terms of weights not exceeding r in which one composes as usual but neglects terms of weight larger than r. Given a differential operator $\mathfrak{D}$ on X with linear coefficients the commutator $\mathfrak{D}\sigma-\sigma\mathfrak{D}$ of $\mathfrak{D}$ and an element σ of $\mathfrak{A}$ is in $\mathfrak{A}$. Furthermore, if σ is in $\mathfrak{F}^{r+1}(X)$ and $\mathfrak{D}$ has no term of order zero then $\mathfrak{D}\sigma-\sigma\mathfrak{D}$ is in $\mathfrak{F}^{r+1}(X)$. Thus this commutator defines a derivation in $\mathfrak{A}^r(X)$ which we will denote also by $\mathfrak{D}$.

The space R^n can be regarded as a vector space T and Fourier transforms of functions on R^n can be regarded as functions on the dual T^* of T. Given an operator A in $\mathcal{S}^r(R^n)$, we associate with A a function $\sigma(A)(x)$ on R^n with values in $\mathfrak{A}^r(T^*)$ in the following fashion. If A is represented as in (20) and $h(x, z)=\sum_0^r h_j(x, z)$, where h_0 is the symbol of H_0 and h_j is the functions associated with H_j as in (20), then $h(x, z)$ is a function defined on the cartesian product of R^n and T^*. With each linear function on R^n or T there is associated an element v of T^*, and with an element v of T^* we associate the differential operator

$$(2\pi i)^{-1} \lim_{t\to 0} \frac{1}{t} f(x + vt) - f(x)]$$

on functions on T^*. This mapping of linear functions on T into differential operators with constant coefficients on T^* can be extended to a homomorphism of polynomial functions on T into differential operators with constant coefficients on T^*. Now we construct $\sigma(A)(x_0)$ as follows: we expand $h(x_0+u, z)$, $u\in T$, in formal power series in u, we neglect all terms of degree larger than r and replace each polynomial in u by its corresponding differential operator on T^*, keeping the coefficients—which are functions of z, that is, functions on T^*—on the right. This yields an operator in $\mathfrak{A}(T^*)$. Its image in $\mathfrak{A}^r(T^*)$ will be $\sigma(A)(x_0)$ by definition.

Thus the symbol of an operator in $\mathcal{S}^r(R^n)$ is a function on R^n with values in $\mathfrak{A}^r(T^*)$. The basic property of the symbol is that the mapping $A\to\sigma(A)$ is linear and multiplicative, i.e., it is a homomorphism of $\mathcal{S}^r(R^n)$ into the algebra of functions with values in $\mathfrak{A}^r(T^*)$. Furthermore, $\sigma(A^*)=\sigma(A)^*$ where A^* is the adjoint, or conjugate transposed of A, and $\sigma(A)^*$ is the formal adjoint of $\sigma(A)$, and $\sigma(A)=0$ if and only if $A\in\mathfrak{J}^{r+1}(R^n)$. Not every function on R^n with values in $\mathfrak{A}^r(T^*)$ is the symbol of an operator in $\mathfrak{A}^r(R^n)$. Aside from the obvious differentiability and boundedness conditions there are certain integrability conditions to be satisfied. Given an element u of T we associate with it the function $L_u(v)=(2\pi i)(u, v)$ on T^*. This linear function on T^* can be regarded as a differential operator of order zero with linear coefficients acting on functions on T^*, and then the condition for a function F on R^n with values in $\mathfrak{A}^r(T^*)$ to be a symbol is that

$$(dF, u) + L_uF - FL_u \equiv 0 \pmod{\mathfrak{F}^r(T^*)} \tag{22}$$

everywhere in R^n for all u in T, where (dF, u) is the differential of F evaluated at u.

An important invariance property of the symbol is the following: if A and B are as in (21) and δ has a contact of order $r+1$ with the identity diffeomorphism at a point x_0 of $V\cap W$, then if $\phi_1=\phi_2=1$ near x_0, then

$$\sigma(A)(x_0) = \sigma(B)(x_0). \tag{23}$$

We turn now to operators on manifolds. Referring to the definition of $\mathcal{S}_m(\mathfrak{M})$ in §7 we will say that an operator A is in $\mathcal{S}^r(\mathfrak{M})$ if it belongs to $\mathcal{S}_m(\mathfrak{M})$ for all m and satisfies the additional conditions:

(i′) the operator N_1AN_2 in (i) maps $L^p_m(\mathfrak{M})$ continuously into $L^p_{m+r+1}(\mathfrak{M})$ for $1<p<\infty$ and $-\infty<m<\infty$;

(ii′) the operator B in (ii) belongs to $\mathcal{S}^r(R^n)$.

Let now x be a point of $\mathfrak{M}$ and let J_x^{r+1} be the space of jets of order $(r+1)$ of local diffeomorphisms of the tangent space T_x into $\mathfrak{M}$, mapping the zero element of T_x into the point x of $\mathfrak{M}$. Let $J^{r+1}(\mathfrak{M})$ be the bundle over $\mathfrak{M}$ with fibers J_x^{r+1} and $T^*(\mathfrak{M})$ the cotangent bundle over $\mathfrak{M}$. Given a point p in $J^{r+1}(\mathfrak{M})$ above the point x in $\mathfrak{M}$, by definition there is associated with p a class of diffeomorphisms of neighborhoods of the zero in T_x into neighborhoods of x in $\mathfrak{M}$. Any two diffeomorphisms in the class have a contact of order $(r+1)$ at the zero element of T_x. Given a diffeomorphism δ in the class, an operator A in $\mathcal{S}^r(\mathfrak{M})$ and an infinitely differentiable function ϕ on $\mathfrak{M}$ with sufficiently small support containing x, there is an operator B in $\mathcal{S}^r(T_x)$ such that

$$(NANf) \circ \delta = B(f \circ \delta)$$

for every f with sufficiently small support containing x, N being the operator multiplication by ϕ. If $\phi = 1$ near x, then according to (23), $\sigma(B)(0)$ is independent of the choice of δ. Thus, for a given operator A we obtain a function $\sigma(A)$ which assigns to a point p of $J^{r+1}(\mathfrak{M})$ above the point x in $\mathfrak{M}$ an element of $\mathcal{A}^r(T_x^*)$. This function we shall call the symbol of A, and it has the required properties, namely, the mapping $A \to \sigma(A)$ is a homomorphism of the algebra $\mathcal{S}^r(\mathfrak{M})$. Furthermore, $\sigma(A) = 0$ if and only if A maps $L_m^p(\mathfrak{M})$ continuously into $L_{m+r+1}^p(\mathfrak{M})$ for $1 < p < \infty$ and all m.

As in the case of Euclidean space, not every function of the kind just described is the symbol of an operator. There are integrability and certain invariance conditions to be satisfied. In any local coordinate system the integrability condition can be expressed as in (22). The invariance conditions refer, roughly speaking, to the behaviour of the symbol under coordinate transformations. Let J_x^{r+1} be the fiber of $J^{r+1}(\mathfrak{M})$ above the point x of $\mathfrak{M}$, and let p be a point in J_x^{r+1}. Then there is a canonical linear map associating with each vector u tangent to J_x^{r+1} at p a differential operator with linear coefficients and without term of order zero on T_x^*. As we saw above, with such an operator there is associated a derivation $\mathfrak{D}_u$ of $\mathcal{A}^r(T_x^*)$. If F is a function as described above, its restriction to J_x^{r+1} is a function with values in $\mathcal{A}^r(T_x^*)$ and the invariance condition is that

$$(dF, u) + \mathfrak{D}_u(F) = 0$$

for all u and everywhere in $J^{r+1}(\mathfrak{M})$, where (dF, u) is the differential of the function F on J_x^{r+1} evaluated at u. On the other hand, this plus the integrability and the obvious differentiability conditions are sufficient conditions for F to be the symbol of an operator in $\mathcal{S}^r(\mathfrak{M})$.

9. Applications to the theory of functions of real variables. As was pointed out in §2, singular integral operators preserve many local properties of functions. This fact can be used advantageously in the study of such properties, and it was indeed this study that motivated much of the development of the theory of singular integrals. Let us describe briefly a few results in this topic. Consider the spaces T_u^p introduced at the end of §2, and let $T_u^p(x)$ be the space obtained by replacing the origin by x in the definition of T_u^p. Suppose $Lf=g$ is an elliptic, possibly over-determined system of partial differential equations with smooth coefficients such as, for example, $\Delta f=g$ or grad $f=g$. Suppose that u is not an integer, that $1<p<\infty$ and that $f\in L^p$. Then if g is in $T_u^p(x)$, f is in $T_{u+m}^p(x)$, m being the order of L. Furthermore, derivatives of f of order j, $j\leqq m$, are in $T_{u+m-j}^p(x)$. If g is in $T_u^p(x)$ for all x in a closed set C and its norm as an element of that space is a bounded function of x, then f coincides on C with a function having continuous derivatives of all orders less than $m+u$ and with highest order derivatives satisfying a uniform Hölder condition of exponent equal to the fractional part of u. If u is an integer and g belongs to $T_u^p(x)$ for all x in a set of positive measure, then the derivatives of order j of f, $0\leqq j\leqq m$, belong to $T_{u+m-j}^p(x)$ for almost all x in the set.

10. A priori inequalities, initial value problems and unique continuation of solutions of partial differential equations. In this section we will describe how the theory of singular integrals can be applied to the solution of problems in these areas. First let us explain what an *a priori* inequality is and why such inequalities play an important role in the theory of partial differential equations. Let L be a linear and possibly unbounded operator from one Banach space into another, and let L have a densely defined adjoint L^*. Then, if L satisfies the inequality $\|Lu\|>c\|u\|$ for some $c>0$ and every u in its domain, L^* is onto. This is not difficult to show and it means that the equation $L^*u=v$ is solvable for every given v. If L is a partial differential operator, the preceding inequality is what is usually called an *a priori* inequality. Suppose now that L is a linear partial differential operator on R^{n+1} with bounded and boundedly differentiable coefficients. Let us denote points in R^{n+1} by (x, t) where x is in R^n and t is in R. Suppose furthermore that the coefficient of D^m in L is 1, D denoting differentiation with respect to t and m being the order of L. Then L can be expressed as

$$L = D^m + \sum_1^m L_j D^{m-j},$$

where L_j is a differential operator on R^n of order j depending on the parameter t. If we represent L_j as $L_j = A_j(I+\Lambda)^j$, where A_j is an operator in $\mathcal{S}_r(R^n)$, $r=1$, depending on the parameter t, and substitute above we obtain

$$L = D^m + \sum_1^m A_j(I+\Lambda)^j D^{m-j}.$$

Given an equation $Lf = g$ we introduce new dependent variables

$$u_j = (I+\Lambda)^{m-j}D^{j-1}f,\ 1 \leqq j \leqq m;\quad v_j = 0,\ 1 \leqq j \leqq m-1;\quad v_m = g$$

and the equation $Lf = g$ combined with the equations

$$Du_j - (I+\Lambda)u_{j+1} = 0$$

becomes a system of the form

$$Du + H(I+\Lambda)u = v,$$

where $u = (u_1, \cdots, u_m)$, $v = (v_1, \cdots, v_m)$ and H is an $m \times m$ matrix of operators in $\mathcal{S}_r(R^n)$ depending on the parameter t. Suppose now that the complex characteristics of L are simple in the direction dual to $t=0$, then the eigenvalues of the matrix of symbols $\sigma(H)$ are distinct and there exists a positive hermitian matrix $\sigma(N)$ such that $\sigma(N)^{-1}\sigma(H)\sigma(N)$ is normal. If in addition the eigenvalues of $\sigma(H)$ remain uniformly apart, then $\sigma(N)$ satisfies the smoothness conditions for it to be the matrix of symbols of an invertible matrix N of operators in $\mathcal{S}_r(R^n)$ depending smoothly on the parameter t. Let us rename u and v respectively u' and v' and introduce again new dependent variables $u = N^{-1}u'$, $v = N^{-1}v'$. Then, using the fact that $N\Lambda - \Lambda N$ is bounded in $L^2(R^n)$,

$$L_1 u = Du + K_1(I+\Lambda)u + K_2 u = v \tag{24}$$

where K_1 is in $\mathcal{S}_r(R^n)$ and K_2 is a bounded operator on $L^2(R^n)$ and they both depend on the parameter t. Furthermore $\sigma(K_1)$ is a normal matrix. This makes the equation easier to manipulate and *a priori* inequalities can easily be derived from it. Let us assume for example that L is totally hyperbolic, then the eigenvalues of $i\sigma(H)$ are real and $i\sigma(K_1)$ is hermitian. Let us consider functions $u(x, t)$ in the strip $0 \leqq t \leqq 1$ vanishing at $t=0$. Given two such vector valued functions let

$$(u, v) = \int_{R^n} \sum_1^m u_j(x, t)\bar{v}_j(x, t)\, dx$$

and let $\|u\|$ and $\|v\|$ denote the norms of u and v in L^2 of the strip. Then from (24), forming the inner product with u on the right and on the left, and multiplying by e^{-at} we obtain

$$D[(u, u)e^{-at}] + [(K_1u, u) + (u, K_1u) + ((K_1\Lambda + \Lambda \overset{*}{K_1})(u, u)) + (K_2u, u) + (u, K_2u) + a(u, u)]e^{-at} = [(u, v) + (v, u)]e^{-at}.$$

Since $i\sigma(K_1)$ is hermitian, we have $\sigma(K_1) = -\sigma(K_1^*)$ and therefore the operator $K\Lambda + \Lambda K_1^*$ is bounded in $L^2(R^n)$ uniformly in t. Furthermore, the same is true of K_1 and K_2 and therefore the sum of the first five terms in the square brackets above, which is real, is not less than $-c(u, u)$. Thus, integrating between 0 and 1 and dropping the integrated term, which is positive, we obtain

$$(a - c)\|u\|^2 e^{-a} \leqq 2\|u\| \, \|v\|,$$

or

$$\|L_1u\| \geqq (1/2)(a - c)e^{-a}\|u\|.$$

This inequality can be translated back into an inequality between f and its derivatives up to order $m-1$ and g, namely the so-called energy inequality for hyperbolic equations.

More elaborate but equally elementary calculations yield inequalities under various assumptions on L, from which existence and uniqueness theorems can be derived.

11. Applications to topology. Atiyah and Singer have shown that a number of topological invariants of a manifold are given by the index of certain systems of differential or singular integral equations on the manifold. They also have given a formula for the index. Very recently the calculation of the index has been simplified to a point where only little and mostly elementary topology is needed. I will outline here this simplified method of calculation.

All manifolds that we shall consider from now on will be assumed to be Riemannian. Given an $m \times m$ matrix A of singular integral operators on a manifold $\mathfrak{M}$, of class $\bar{\mathcal{S}}_\infty(\mathfrak{M})$, we will regard A as an operator acting on vector valued functions on $\mathfrak{M}$ whose components belong to one of the function spaces on which we know A to operate continuously, say for example $L_r^p(\mathfrak{M})$, $1 < p < \infty$. The corresponding matrix of symbols $\sigma(A)$ will be regarded as a matrix valued function on the cotangent sphere bundle $S^*(\mathfrak{M})$, rather than on $T^*(\mathfrak{M})$. All properties of the symbols of singular integral operators are shared by these matrix symbols, except of course for the commutativity of multiplication. A matrix A of singular integral operators is called elliptic if the determinant of $\sigma(A)$ does not vanish. If this is the case, then there exists another matrix A' such that $\sigma(A)\sigma(A')$ is the identity matrix and therefore $AA'=I+S$, where I is the identity operator and S is smoothing and completely continuous. Since S is completely continuous $I+S$ is a Fredholm operator and therefore it has a finite-dimensional null space and closed range of finite codimension. Thus A has closed range of finite codimension, and A' has a finite-dimensional null space. But evidently, A and A' are interchangeable here and we conclude that A also has finite-dimensional null space. The index $i(A)$ is defined as the dimension of the null space of A minus the codimension of its range, and it is this integer $i(A)$ that we intend to calculate.

Let us begin discussing the properties of the index. From the general theory of operators we know that $i(A)$, whenever it is defined, depends continuously on A with respect to the uniform operator topology. Furthermore, if $i(A)$ and $i(B)$ are defined so are $i(AB)$ and $i(A+S)$ for every completely continuous S, and $i(AB)=i(A)+i(B)$, $i(A+S)=i(A)$. If A and B are two elliptic matrices of singular integral operators such that $\sigma(A)=\sigma(B)$, then $A=B+S$ where S is smoothing and completely continuous and therefore $i(A)=i(B)$, which shows that $i(A)$ depends only on $\sigma(A)$. Since the symbol mapping has continuous right inverse τ, it follows that $i(A)$ depends continuously on $\sigma(A)$ with respect to the topology in the space of symbol functions we described in §7. Now the values of $i(A)$ are integers and therefore $i(A)$ is invariant under sufficiently differentiable homotopies of $\sigma(A)$; but if any two symbol functions are homotopic in the ordinary sense then they are also homotopic with respect to the topology of symbol functions described in §7. Therefore $i(A)$ is invariant under ordinary homotopies of $\sigma(A)$. Summarizing, if the space of nonsingular matrices of symbols with the operation of pointwise multiplication and the topology of uniform convergence is regarded as topological group, the index is a continuous homomorphism of this group into the additive group of the integers.

An additional important property of the index is the following. Suppose that two manifolds $\mathfrak{M}_1$ and $\mathfrak{M}_2$ are identified along a pair of their open subsets $V_1=V_2=V$. Suppose that A_1 is an elliptic matrix of singular integral operators on $\mathfrak{M}_1$ such that $\sigma(A_1)=I$ at all points of $S^*(\mathfrak{M}_1)$ above points of $\mathfrak{M}_1$ outside a compact subset C of V. Then if ϕ is an infinitely differentiable function on $\mathfrak{M}_1$ such that $\phi=1$ on C and $\phi=0$ in a neighborhood of the complement of V, we can regard the operator $\phi A_1\phi$ as an operator on $\mathfrak{M}_2$. Let now ψ on $\mathfrak{M}_2$ vanish in the complement of $V_2=V$ and coincide with ϕ on V, and let A_2 be the operator on $\mathfrak{M}_2$ defined by $(1-\psi^2)I+\phi A_1\phi$. Then $i(A_1)=i(A_2)$.

This important property permits us to reduce the calculation of the index on one manifold to the corresponding calculation on another. Its proof is very simple. In fact, consider the operator $A_1' = (1-\phi^2)I + \phi A_1 \phi$ on $\mathfrak{M}_1$. Then, clearly $\sigma(A_1') = \sigma(A_1)$ and therefore $i(A_1') = i(A_1)$. Furthermore, as is readily seen, every function in the null space of A_1', or orthogonal to its range, has support in V. Clearly the same is true of A_2. Since A_1' and A_2 coincide in V we find that $i(A_1') = i(A_2)$ and the desired conclusion follows.

We pass now to the calculation of the index. It is accomplished in two steps. The first employs a construction due to Atiyah and Singer and consists in reducing the problem to the case of an odd-dimensional sphere. The second consists in finding the index in this special case.

Given an elliptic matrix A of singular integral operator on a manifold $\mathfrak{M}$, we embed $\mathfrak{M}$ in an odd-dimensional sphere and consider a cylindrical neighborhood V of the image of $\mathfrak{M}$. This neighborhood V can evidently be regarded as an open subset of a sphere bundle $\mathfrak{B}$ over $\mathfrak{M}$, in which $\mathfrak{M}$ is embedded as a cross section and V is a cylindrical neighborhood of this cross section. We will construct an elliptic $\overline{A}$ of singular integral operators on $\mathfrak{B}$ such that $\sigma(\overline{A}) = I$ outside a compact subset of V and for which $i(\overline{A}) = 2i(A)$, and by the device described above the calculation of $i(\overline{A})$ will be reduced to the case of an odd-dimensional sphere. Let S be the fiber of $\mathfrak{B}$, p its projection and S_x the fiber above the point x of $\mathfrak{M}$. There exists a matrix B of operators on $\mathfrak{B}$ with the following properties: if R_x is the operator restriction to S_x, then $R_x B = B_x R_x$, where B_x is an elliptic matrix of singular integral operators on S_x such that $\sigma(B_x) = I$ outside V, B_x is onto and the nullspace of B_x is two-dimensional and consists of functions depending only on the distance to x. The construction of B is not difficult but its description would take too long. With B we construct another operator $B' = (I+\Lambda_1)^r B$, where Λ_1 is an operator on $\mathfrak{B}$ such that $R_x \Lambda_1 = \Lambda_x R_x$, Λ_x, being a Λ-operator on S_x. Clearly, B' should be regarded as an operator from $L_r^p(\mathfrak{B})$ into $L^p(\mathfrak{B})$ and every function in the null space of B' depends only on the distance to $\mathfrak{M}$. Correspondingly we construct an operator A' on $\mathfrak{B}$ associated with A. We can regard $\mathfrak{B}$ as a coordinate bundle with the rotations of S about x_0 as group. If V_j are coordinate neighborhoods of $\mathfrak{B}$ and ψ_j are the corresponding coordinate functions, we let $\sum \phi_j^2 = 1$ be a partition of unity on $\mathfrak{M}$ such that ϕ_j is supported in V_j. On $V_j \times S$ we define an operator A_j as follows: we let Λ_2 be a Λ operator on $\mathfrak{M}$ and for a function $f(x, y)$ $x \in V_j$, $y \in S$, we set $A_j f = \phi_j (I+\Lambda_2)^r A \phi_j f$. Then we let A_j' be the operator on $p^{-1}(V_j)$ which is the image of A_j under ψ_j. Now A_j' can be extended in the obvious fashion to operate on all functions on $\mathcal{B}$ and we finally define $A' = \sum A_j'$. The operator A' has the following properties: if f is a numerical function on $\mathcal{B}$ which merely depends on the distance to the cross section $\mathfrak{M}$, g[3] then $A'fg = fA'g$; furthermore, if $\tilde{A} = \sum \phi_j(I+\Lambda_2)^r A\phi_j$ and g is a function on $\mathfrak{M}$ then $p^{-1}(\tilde{A}g) = A'p^{-1}(g)$. Suppose now that A and B are $m \times m$ and $n \times n$ matrices respectively, then we denote the corresponding identity matrices by I_m and I_n; further, we let Λ be a Λ operator on $\mathcal{B}$ and define the matrix of operators

$$A \# B = (I + \Lambda)^{-r} \begin{pmatrix} A' \otimes I_n & -I_m \otimes (B')^* \\ I_m \otimes B' & (A')^* \otimes I_n \end{pmatrix}$$

on $\mathcal{B}$.[3] It is not difficult to show that the operators in this matrix are integral operators in $\bar{\mathcal{S}}_\infty(\mathcal{B})$, provided that r is larger than twice the dimensions of $\mathcal{B}$. This has to be done merely locally and the expansion in series (15) can be used advantageously for this purpose. Furthermore, the matrix is elliptic and given the properties of A' and B' one can show without difficulty that $i(A \# B) = i(\tilde{A})i(B)$. But $\tilde{A}$ is of the form $\tilde{A} = (I+\Lambda_2)^r C$ where $\sigma(C) = \sigma(A)$. Thus $i(\tilde{A}) = i(A)$ and $i(A \# B) = i(A)i(B)$.[4]

The symbol of $A \# B$ does not have the property of being the identity in a neighborhood of the cross section $\mathfrak{M}$. For this purpose we construct $A \# I$ obtained by replacing B by the corresponding identity matrix in the construction above. This operator has the property that $\sigma(A \# B) = \sigma(I \# B)$ in a neighborhood of the cross section $\mathfrak{M}$. Consequently, if $\bar{A}$ is any operator on $\mathcal{B}$ such that $\sigma(\bar{A}) = \sigma(A \# B)\sigma(A \# I)^{-1}$ its symbol will have the required property. Furthermore, since $i(A \# I) = i(I)i(A) = 0$ we will have $i(\bar{A}) = i(A \# B) = i(A)i(B) = 2i(A)$.

This completes the reduction of the calculation of the index to the case of an odd dimensional sphere in the case where $\mathfrak{M}$ is even-dimensional and orientable. If $\mathfrak{M}$ is odd-dimensional and orientable, we construct by the method above a matrix of singular integral operators on the cartesian product of $\mathfrak{M}$ and a one-sphere with index $2i(A)$, and this puts us back in the case of an even-dimensional manifold.

[3] The adjoints here are taken with respect to the density induced by the metric of $\mathcal{B}$.

[4] In calculating these indices one must use the fact that the nullspaces of our operators always consist of infinitely differentiable functions and therefore are independent of the functional space on which the operator is assumed to operate. In fact, if $\sigma(A_1)\sigma(A) = I$ then $A_1 A = I + S$, where S is smoothing of class m for all m. Thus if $Af = 0$ then $f = -Sf$, and if f belongs to L_r^p, it belongs to L_{r+1}^p, and consequently, it is infinitely differentiable.

We pass now to the calculation of the index of elliptic matrices of singular integral operators on odd-dimensional spheres. We begin with some observations. If σ is a nonsingular symbol matrix we can write $\sigma = uh$, where u is unitary and h is positive hermitian. Since the space of positive hermitian matrices is contractible, h is homotopic to a constant and consequently $i(h)=0$. Thus we can restrict ourselves to consider symbols whose values are unitary matrices. Functions with unitary matrices as values will henceforth be denoted by $f, f_1, \cdots$, etc. It is well known that functions on sphere S with values in the unitary group U_k have the property that if $[f]$ denotes the homotopy class of f then $[f_1f_2]=[f_1]+[f_2]$. We shall regard the group of unitary matrices U_k, $k=1, 2, \cdots$, as embedded in U_{k+1} in the usual fashion. The natural projection of U_k into the $2k-1$ dimensional sphere $S_{2k-1}=U_k/U_{k-1}$ will be denoted by p_k.

Let M denote a manifold of dimension m and let f be a function on M into U_n. If $m<2n-1$, then $p_n \circ f$ maps M into a sphere of dimension larger than m and consequently it is homotopic to a constant mapping of M into, say, $p_n(U_{n-1})$. Thus by the covering homotopy theorem it follows that f is homotopic to a mapping of M into U_{n-1}. Iterating this argument we find that f is homotopic to a mapping of M into U_k, k being the smallest integer such that $2k \geqq m$.

Suppose now that M is a sphere S_m and that $m \neq 2n-1$. If f is again a function on S_m into U_n and $[p_n \circ f]$ is the homotopy class of $p_n \circ f$, on account of the finiteness of $\pi_m(S_{2n-1})$ there exists an integer r such that $r[p_n \circ f]=0$. But $r[f]=[f^r]$ and consequently $[p_n \circ f^r] = r[p_n \circ f]=0$, and $p_n \circ f^r$ is homotopic to a constant. Using again the covering homotopy theorem we conclude that f^r is homotopic to a mapping of S_m into U_{n-1}. Iterating this argument we find that if m is even, or if $m>2n-1$, then f has a power which is homotopic to a constant.

Let now the dimension m of M be equal to $2n-1$. Take on S_{2n-1} a differential form $\bar{\omega}$ of degree $2n-1$, invariant under rotations and of integral equal to 1, and let $\omega = p_n^* \bar{\omega}$ be its image on U_n under p_n. Then if f is a function on S_m into U_n we define

$$j(f) = \int_M f^*\omega = \int_M (p_n \circ f)^* \bar{\omega}.$$

This is clearly the degree of the mapping $p_n \circ f$ of M into S_{2n-1}. Thus if $j(f)=0$ then $p_n \circ f$ is homotopic to a constant and f is homotopic to a mapping of M into U_{n-1}. Another important property of $j(f)$ is that $j(f_1f_2)=j(f_1)+j(f_2)$. To see this we let D_1 and D_2 be two spherical open subsets of M with disjoint closures and given f_1 we let h be a mapping of M into S_{2n-1} mapping the complement of D_1 into the point $p_n(U_{n-1})$ and having the same degree as $p_n \circ f_1$. Then $p_n \circ f_1$ is homotopic to h and by the covering homotopy theorem f_1 is homotopic to a mapping f_1' such that $p_n \circ f_1' = h$. This property of f_1' implies that f_1' maps the complement of D_1 into U_{n-1}. Now it is readily seen that f_1' is in turn homotopic to a mapping f_1'' coinciding with f' on D_1 mapping the complement of D_1 into U_{n-1} and mapping D_2 into the identity element of U_n. Similarly, given f_2 we construct f_2'' homotopic to f_2, mapping the complement of D_2 into U_{n-1} and D_1 into the identity element of U_n. But then $p_n \circ (f_1''f_2'')$ coincides with $p_n \circ f_1''$ in D_1, with $p_n \circ f_2''$ in D_2 and is constant elsewhere. This implies that

$$[p_n \circ (f_1''f_2'')]^*\bar{\omega} = (p_n \circ f_1)^*\bar{\omega} + (p_n \circ f_2)^*\bar{\omega},$$

and from this it follows that

$$j(f_1f_2) = j(f_1''f_2'') = j(f_1'') + j(f_2'') = j(f_1) + j(f_2).$$

Now we will extend the definition of j to functions on the manifold M of dimension $m = 2n-1$ into U_k, $k > n$. For this purpose we construct in U_k/U_{n-1} a closed differential form $\bar{\omega}_k$ such that if e is the injection of U_n/U_{n-1} into U_k/U_{n-1} then $e^*\bar{\omega}_k = \bar{\omega}$. This we do as follows. We regard U_k/U_{n-1} as a bundle over U_k/U_n with fiber $U_n/U_{n-1} = S_{2n-1}$.

The top dimensional homology class α in U_n/U_{n-1} maps into an element $\bar{\alpha}$ of $H_{2n-1}(U_k/U_{n-1})$ which is not zero. (This follows without difficulty from the homotopy sequence of U_k/U_{n-1} and the isomorphism theorem of Hurewicz.) Consequently there is an element β of $H^{2n-1}(U_k/U_{n-1})$ which evaluated at $\bar{\alpha}$ is not zero. Integrating β over U_k we obtain the desired $\bar{\omega}_k$. Now if f is a function on M into U_k and q is the natural projection of U_k into U_k/U_{n-1} we define $\omega_k = q^*\bar{\omega}_k$ and

$$j(f) = \int_M f^*\omega_k = \int_M (q \circ f)^*\bar{\omega}_k.$$

This definition clearly coincides with the previous one in the case when f happens to have values in U_n. Since $j(f)$ is invariant under homotopies of f and every such f can be deformed to take values in U_n, the relation $j(f_1f_2) = j(f_1) + j(f_2)$ still holds in this more general situation.

We are now ready to derive the formula for the index. Let S be a sphere of odd dimension equal to n, and let $S^*(S)$ be the cotangent bundle over S and p its projection on S. Consider a symbol function f of $S^*(S)$ into U_k, $k \geqq n$, for which $j(f) = 0$. We will show that in this case $i(f) = 0$. Let x_0 be a point of S. Since $p^{-1}(x_0)$ is an even-dimensional sphere there is a power f^r of f which is homotopic to a constant on $p^{-1}(x_0)$ and therefore there exists an f_1 homotopic to f^r such that f_1 is constant in the inverse image $p^{-1}(D)$ of an open disc in S with center at x_0. If D' and $\overline{D}$ denote the complement and the closure of D respectively, and for $x\in\overline{D}$ we identify $p^{-1}(x)$ with x, then $p^{-1}(D')$ becomes a sphere S' of dimension $2n-1$, $S^*(S)$ becomes $S'\cup\overline{D}$ and f_1 becomes a function on $S'\cup\overline{D}$, and we have

$$\int_{S'} f_{1,\omega_k}^{*} = \int_{S^*(S)} f_{1,\omega_k}^{*} = j(f_1) = j(f^r) = rj(f) = 0.$$

This means that f_1 is homotopic on S' to a function with values in U_{n-1} and consequently there exists an integer s such that F_1^s is homotopic to a constant on S'. Now this homotopy can be extended to $S'\cup\overline{D}$ and, as it is readily seen, this implies that f_1^s is homotopic on $S^*(S)$ to a function f_2 of the form $f_2=g(p(\cdot))$, where g is a function on S with values in U_k. Consider now the operator A on S which consists in operating on vector valued functions on S by means of the matrix valued function g. Then evidently $i(A)=0$, and since $\sigma(A)=g(p(\cdot))=f_2$, it follows that $i(f^{rs})=i(f_1^s)=i(f_2)=0$, and consequently $i(f)=0$. But i is a continuous homomorphism of the group symbol functions on $S^*(S)$ with values in U_k into the additive group of integers, and so is j. Since $j=0$ implies $i=0$, it follows that there exists a constant c such that $i(f)=cj(f)$. This constant can be calculated if $j(f)$ is calculated for an f for which $i(f)$ is known and turns out to be $1/(n-1)!$ if proper orientations are chosen. Thus we have

$$(n-1)!\, i(f) = j(f) = \int_{S^*(S)} f^*\omega_k.$$

This formula can be extended to symbol functions f whose values are not necessarily unitary. For this purpose one must extend ω_k to a closed form on the group of nonsingular $k\times k$ matrices. Since this group is the cartesian product of U_k and the space H of positive definite hermitian $k\times k$ matrices this is accomplished by the form $(\omega_k\otimes 1)$ on $U_k\times H$.

The index theorem of Atiyah and Singer refers to singular integral operators on sections of a vector bundle over a manifold. It can be easily shown however that such an operator is equivalent to a system of the kind we have been discussing. The calculation of its index can therefore be carried out by the method described above.

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