INTEGRAL REPRESENTATION OF MULTILINEAR CONTINUOUS OPERATORS FROM THE SPACE OF LEBESGUE-BOCHNER SUMMABLE FUNCTIONS INTO ANY BANACH SPACE

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In [3] was found a representation of linear continuous operators h from the space $L(v, Y)$ of Lebesgue-Bochner summable functions into any Banach space W. The above result was generalized to the case of the space $L_p(v, Y)$ in [4].

In this paper it will be shown that by means of the integral $\int u(f_1, \cdots, f_k, d\mu)$ one can represent multilinear continuous operators from the space

$$L(v_1, Y_1) \times \cdots \times L(v_k, Y_k)$$

into any Banach space. We use the notation of [2] and [8].

Some representations of linear continuous functionals and operators on the space $L_p(v, Y)$ were obtained by Dieudonné [11] and Dinculeanu [12].

The problem of representation of linear continuous operators from the space $C(X, Y)$ of all continuous functions on a compact space X into a Banach space Y, or more generally into a locally convex space, is closely related to the above problem. The classical result of Riesz [24] was generalized by Dunford and Schwartz [15], Foias and Singer [17], Dinculeanu [13], and Swong [28]. The most general results in this field belong to Dinculeanu [13] and Swong [28]. Some new proofs of the results of Dinculeanu [13] one can find in Bogdanowicz [7].

Let (X, V, v) be the product of volume spaces (X_j, V_j, v_j) $(j=1, \cdots, k)$. Denote by H the space of all multilinear continuous operators from the product of the spaces $L(v_j, Y_j)$ $(j=1, \cdots, k)$ into the Banach space W.

THEOREM 1. *To every k-linear continuous operator h from the product of the spaces* $L(v_j, Y_j)$ $(j=1, \cdots, k)$ *into the Banach space W corresponds a unique vector volume* $\mu \in M(v, W_0)$ *where* $W_0 = L(Y_1, \cdots, Y_k; W)$ *such that*

$$h(f_1, \cdots, f_k) = \int u(f_1, \cdots, f_k, d\mu)$$

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for all $f_j \in L(v_j, Y_j)$ $(j=1, \cdots, k)$, *where*

$$u(y_1, \cdots, y_k, w) = w(y_1, \cdots, y_k) \quad \text{for all } y_j \in Y_j,\ w \in W_0.$$

This correspondence establishes an isometry and isomorphism of the space H *and the space* $M(v, W_0)$.

We have the following theorem characterizing pointwise convergence.

THEOREM 2. *Let* $h_n \in H$ *and let* $\mu_n \in M(v, W_0)$ *be the corresponding sequence of vector volumes* $(n=0, 1, 2, \cdots)$. *Then* $u_n(f_1, \cdots, f_k)$ *converges to* $u_0(f_1, \cdots, f_k)$ *for all* $f_j \in L(v_j, Y_j)$ *if and only if the sequence* μ_n *is bounded and* $\mu_n(A)(y_1, \cdots, y_k) \to \mu(A)(y_1, \cdots, y_k)$ *for all* $A \in V$ *and* $y_j \in Y_j$ $(j=1, \cdots, k)$.

THEOREM 3. *Let* (X, V, v) *be the product of sigma-finite volume spaces* (X_j, V_j, v_j). *Let* h *be a* k*-linear continuous operator from the product of the spaces* $L(v_j, C)$ $(j=1, \cdots, k)$ *into a Banach space* Y, *where* C *denotes the space of complex numbers. Then if* Y *is a separable dual or reflexive space then there exists a function* g *from* X *into* Y *summable on every set* $A \in V$ *and such that*

$$h(f_1, \cdots, f_k) = \int f_1(x_1) \cdots f_k(x_k) g(x_1, \cdots, x_k)\, dv$$

for all $f_i \in L(v_j, C)$ *and*

$$\|h\| = \operatorname{ess\,sup}\{ |g(x)| : x \in X\}.$$

Let (X, V, v) be a volume space and denote by (X_n, V_n, v_n) $(n=1, 2, \cdots)$ the sequence of volume spaces which are products of the volume spaces

$$(X_j, V_j, v_j) = (X, V, v) \qquad (j = 1, \cdots, n).$$

Let Y be a reflexive Banach space or a separable dual to a Banach space.

Let D be an open set in the space $L(v, C)$, where C denotes the space of complex numbers.

THEOREM 4. *Let* $h(f)$ *be a Fréchet analytic function from the set* D *into the space* Y. *Then for every* $f_0 \in D$ *there exists a sequence* $g_n(x_1, \cdots, x_n)$ *of symmetric* v_n*-essentially bounded functions from* X_n *into* Y *such that*

$$h(f_0 + f) - h(f_0) = \sum_{n=1}^{\infty} \int f_1(x_1) \cdots f_n(x_n) g(x_1, \cdots x_n)\, dv_n$$

for all f belonging to a sufficiently small neighborhood of f_0.

To prove this theorem it is enough to expand the analytic function in a neighborhood of f into a series of continuous homogeneous polynomials. See for example Hille and Phillips [**18**, p. 769]. Since every continuous homogeneous polynomial is generated by a multilinear continuous operator therefore using Theorem 3 we can get the representation stated in the theorem.

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