NEW PROOF OF A THEOREM OF GAIFMAN AND HALES

BY ROBERT M. SOLOVAY

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For terminology on Boolean algebras see [2], [3]. In particular, a set X generates a complete Boolean algebra $\mathcal{B}$ if $X \subseteq \mathcal{B}$, but for each proper complete subalgebra $\mathcal{B}'$ of $\mathcal{B}$, we have $X \not\subseteq \mathcal{B}'$.

1. THEOREM 1 ([2], [3]). *There are countably generated complete Boolean algebras of arbitrarily high cardinality.*

Our proof, which is much simpler than those of Gaifman and Hales, is inspired by some recent work of Paul Cohen [1]. The connection between forcing and Boolean algebras will be elucidated in [5]. However, our proof will be independent of [1] and [5]. I am grateful to Dana Scott for helpful conversations concerning these proofs.

2. **The example.** Let $\aleph_\tau$ be an infinite cardinal. We identify $\aleph_\tau$ with the set of ordinals of cardinality less than $\aleph_\tau$; we give $\aleph_\tau$ the discrete topology. Let X be the product space $\aleph_\tau^{\aleph_0}$ endowed with the product topology. Thus an element $f \in X$ is a map from $\aleph_0$ into $\aleph_\tau$. Let $U(n, f)$ be

$$\{g \in X: g(m) = f(m) \text{ for } m \leqq n\}. \tag{1}$$

The sets $U(n, f)$ for $n < \omega$ form a neighborhood basis of f.

Let $\mathcal{B}$ be the algebra of regular open sets of X. (An open set U is regular if $U = \operatorname{int}(\operatorname{cl} U)$: Here $\operatorname{int} E$ is the interior of E and $\operatorname{cl} E$ is the closure of E. In particular, an open closed set is regular.) According to [4], $\mathcal{B}$ is complete. Moreover, the Boolean operations $\bigwedge$, $\bigvee$, C (infinite sup, infinite inf, and complement) are given by the following formulas:

$$\bigvee \mathfrak{U} = \operatorname{int}(\operatorname{cl} \bigcup \mathfrak{U}) \qquad (\mathfrak{U} \subseteq \mathcal{B}) \tag{2}$$

$$\bigwedge \mathfrak{U} = \operatorname{int}(\bigcap \mathfrak{U}) \qquad (\phi \neq \mathfrak{U} \subseteq \mathcal{B}) \tag{3}$$

$$CU = \operatorname{int}(X - U) \qquad U \in \mathcal{B}. \tag{4}$$

Using (2) and (3), one sees that $\mathcal{B}$ is generated by the family $\{A_{n,\eta}: n < \omega, \eta < \aleph_\tau\}$, where

$$A_{n,\eta} = \{f \in X: f(n) = \eta\}.$$

Moreover, if $\eta < \eta' < \aleph_\tau$, then the sets $A_{0,\eta}$ and $A_{0,\eta'}$ are distinct. Thus the cardinality of $\mathcal{B}$ is at least $\aleph_\tau$.

3. **The generators.** Let $B_{m,n}=\{f\in X: f(m)\leqq f(n)\}$. The sets $B_{m,n}$ are open closed, and thus determine elements of $\mathfrak{B}$. The following lemma will complete the proof of Theorem 1.

LEMMA 1. *The elements* $\{B_{m,n}|m, n<\omega\}$ *generate* $\mathfrak{B}$.

PROOF. Let $\mathfrak{B}'\subseteq\mathfrak{B}$ be the smallest complete subalgebra containing the elements $B_{m,n}$. Clearly $\{f|f(m)<f(n)\}$ lies in $\mathfrak{B}'$. (It has complement $B_{n,m}$.) To prove Lemma 1, it suffices to show $A_{n,\eta}\in\mathfrak{B}'$ for every $n<\omega$, and $\eta<\aleph_\tau$. (Since $\{A_{n,\eta}\}$ generates $\mathfrak{B}$.)

We prove this by induction on η. Assume then that for $m<\omega$, $\xi<\eta$, we have $A_{m,\xi}\in\mathfrak{B}'$. We shall show that the open closed sets $\{f|f(n)<\eta\}$ and $\{f|f(n)\leqq\eta\}$ lie in $\mathfrak{B}'$. By (3), (4) this will show that $A_{n,\eta}\in\mathfrak{B}'$.

By (2),

$$\{f\,|\,f(n)<\eta\} = \bigvee\nolimits_{\xi<\eta}A_{n,\xi}. \tag{5}$$

Thus $\{f|f(n)<\eta\}$ lies in $\mathfrak{B}'$. It follows that $C_m=\{f|f(m)<f(n)\rightarrow f(m)<\eta\}$ lies in $\mathfrak{B}'$. To complete the proof, it suffices to show

$$\{f\,|\,f(n)\leqq\eta\} = \bigwedge_{m\in\omega} C_m. \tag{6}$$

In view of (2), it suffices to show

$$g(n)\leqq\eta\leftrightarrow g\in\operatorname{int}\Big(\bigcap_{m\in\omega} C_m\Big). \tag{7}$$

The direction $\rightarrow$ of (7) is clear. Suppose now that $g(n)>\eta$, and $U(N, g)\subseteq\bigcap_{m\in\omega}C_m$. We shall get a contradiction. This will complete the proof of (7) and with that Lemma 1.

We may suppose that $N\geqq n$. Let $h:\aleph_0\rightarrow\aleph_\tau$ be defined as follows. For $m\leqq N$, $h(m)=g(m)$; for $m>N$, $h(m)=\eta$. Since $h\in U(N, g)$, we have $h\in C_{N+1}$. But this is absurd since

$$\eta = h(N+1) < h(n) = g(n) \quad\text{and}\quad h(N+1)\geqq\eta.$$

4. We sketch a proof of the following theorem. Let κ be a regular cardinal.

THEOREM 2 ([2], [3]). *There are complete* (κ,∞) *distributive Boolean algebras on* κ *generators of arbitrarily high cardinality.*

PROOF. Let $X=\aleph_\tau^\kappa$ endowed with the κ topology. If $f\in X$, then $\{U(\alpha,f), \alpha<\kappa\}$ forms a neighborhood basis of f; $U(\alpha,f)=\{g\in X|g(\beta)=f(\beta)$ for $\beta\leqq\alpha\}$. Let $\mathfrak{B}$ be the complete Boolean algebra of regular open subsets of X. The proof of [4] Lemma 3, shows that $\mathfrak{B}$ is (κ,∞)

distributive. (In Scott's terminology, $\mathcal{B}$ is (γ, δ) distributive if $\gamma < \kappa$ and δ is a cardinal.) Exactly as in the proof of Theorem 1, one sees that $\mathcal{B}$ has cardinality at least $\aleph_\tau$, and that $\mathcal{B}$ is generated by sets of the form

$$\{f \mid f(\eta) \leqq f(\eta')\}, \quad \textit{where} \quad \eta, \eta' < \kappa.$$

References

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4. D. Scott, *The independence of certain distributive laws in Boolean algebras*, Trans. Amer. Math. Soc. **84** (1957), 258–261.

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University of California, Berkeley