SPIN COBORDISM

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1. **Statements of results.** Ω_*^{Spin}, the Spin cobordism ring, has been studied by many, e.g. Wall [9, p. 294], Milnor [5] and [6], Novikov [7], and P. G. Anderson [3]. In this announcement we describe the additive structure of Ω_*^{Spin}, much of the multiplicative structure, characteristic numbers which determine Ω_*^{Spin}, and other properties.

We first state some technical results. Let $\mathfrak{A}$ denote the mod 2 Steenrod algebra, and let $Q_0 = Sq^1$ and $Q_1 = Sq^3 + Sq^2 Sq^1$. If $a_1, a_2, \cdots, a_r \in \mathfrak{A}$, $\mathfrak{A}(a_1, a_2, \cdots, a_r)$ will denote the left ideal generated by $\{a_i\}$. All cohomology groups will have Z_2 coefficients unless otherwise stated. Let $p: BO\langle n\rangle \to BO$ be the fibre space such that $\pi_i(BO\langle n\rangle) = 0$ for $i < n$ and $p_*: \pi_i(BO\langle n\rangle) \approx \pi_i(BO)$ for $i \geqq n$. The following theorem is due to R. Stong [8].

THEOREM 1.1. *There is an element $\alpha_n \in H^n(BO\langle n\rangle)$ such that the map of $\mathfrak{A}$ into $\overline{H}^*(BO\langle n\rangle)$ given by $a \to a\alpha_n$ defines an isomorphism in dimensions less than $2n$ between $\mathfrak{A}/\mathfrak{A}(Sq^1, Sq^2)$ and $\overline{H}^*(BO\langle n\rangle)$ for $n \equiv 0$ (mod 8) and between $\mathfrak{A}/\mathfrak{A}(Sq^3)$ and $\overline{H}^*(BO\langle n\rangle)$ for $n \equiv 2$(mod 8).*

Let $\xi \in \overline{KO}^0(X)(X)$ be of filtration n [4], that is, ξ is trivial on the $n-1$ skeleton of X. Then there is a map $f_\xi: X \to BO\langle n\rangle$ such that pf_ξ is ξ. Let $[\xi] = \{f_\xi^*(\alpha_n)\} \subset H^n(X)$ for all f_ξ such that $pf_\xi = \xi$.

Let $J = (j_1, \cdots, j_k)$ be a sequence of integers with $j_i > 1$ and $k \geqq 0$. Let $P_J = P_{j_1} \cdots P_{j_k} \in H^{4n(J)}(B\,\text{Spin})$, where $n(J) = \sum j_i$ and P_j is the jth Pontrjagin class. In [2], certain classes $\pi^i \in KO^0(BSO)$ were defined which behave very much like Pontrjagin classes. Under the map $B\,\text{Spin} \to BSO$, π^i maps into a class which we also denote $\pi^i \in KO^0(B\,\text{Spin})$. Let $\pi^J = \pi^{j_1} \cdots \pi^{j_k} \in KO^0(B\,\text{Spin})$. Our main result from KO-theory is the following theorem.

THEOREM 1.2. *The filtration of π^J is $4n(J)$ if $n(J)$ is even, and is $4n(J)-2$ if $n(J)$ is odd. Furthermore, if $n(J)$ is even, there exists $X_J \in H^{4n(J)}(B\,\text{Spin})$ such that $X_J \in [\pi^J]$ and $X_J \equiv P_J$ mod Im Q_0Q_1, and if $n(J)$ is odd, there exists $Y_J \in H^{4n(J)-2}(B\,\text{Spin})$ such that $Y_J \in [\pi^J]$ and $Sq^2 Y_J \equiv P_J$ mod Im Q_0Q_1.*

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Let $M\,\mathrm{Spin}(n)$ denote the universal Thom complex for $\mathrm{Spin}(n)$ bundles, and let $\boldsymbol{M}\,\mathrm{Spin}$ denote the spectrum whose nth term is $M\,\mathrm{Spin}(n)$. We define $\overline{H}^i(\boldsymbol{M}\,\mathrm{Spin})$ to be lim inv $\overline{H}^{n+i}(M\,\mathrm{Spin}(n))$. We let X_J, Y_J, and $P_J \in \overline{H}^*(\boldsymbol{M}\,\mathrm{Spin})$ denote the elements corresponding under the Thom isomorphism to X_J, Y_J, and $P_J \in H^*(B\,\mathrm{Spin})$. Likewise, we let $\pi^J \in \mathrm{Cl}(KO)^0(\boldsymbol{M}\,\mathrm{Spin})$ correspond under the KO-Thom isomorphism to $\pi^J \in KO^0(B\,\mathrm{Spin})$.

Let X and Y be the graded vector spaces over Z_2 generated, respectively, by $\{X_J\}$, $n(J)$ even, and $\{Y_J\}$, $n(J)$ odd. The following theorem gives the structure of $H^*(\boldsymbol{M}\mathrm{Spin})$ as a module over $\mathfrak{A}$.

THEOREM 1.3. *There is a graded vector space Z and an $\mathfrak{A}$-module isomorphism*

$$\theta:(\mathfrak{A}/\mathfrak{A}(Sq^1,Sq^2) \otimes X) \oplus (\mathfrak{A}/\mathfrak{A}(Sq^3) \otimes Y) \oplus (\mathfrak{A} \otimes Z) \to H^*(\boldsymbol{M}\ \mathrm{Spin})$$

such that $\theta(X_J) = X_J$ and $\theta(Y_J) = Y_J$.

Let $\boldsymbol{BO}\langle n\rangle$ denote the Ω-spectrum whose 0th term is $BO\langle n\rangle$. If $n(J)$ is even let $f_J: \boldsymbol{M}\,\mathrm{Spin} \to \boldsymbol{BO}\langle 4n(J)\rangle$ be a map corresponding to π^J such that $f_J^*(\alpha_{4n(J)}) = X_J$ and, if $n(J)$ is odd, let $f_J: \boldsymbol{M}$ Spin $\to \boldsymbol{BO}\langle 4n(J)-2\rangle$ be a map corresponding to π^J such that $f_J^*(\alpha_{4n(J)-2}) = Y_J$. Let $\{Z_i\}$ be a basis for Z and let $f_i: \boldsymbol{M}\,\mathrm{Spin} \to \boldsymbol{K}(Z_2, \dim Z_i)$ be a map corresponding to Z_i ($\boldsymbol{K}(Z_2, n)$ is the spectrum whose 0th term is $K(Z_2, n)$). Let

$$F: \boldsymbol{M}\ \mathrm{Spin} \to \prod_{n(J)\ \mathrm{even}} \boldsymbol{BO}\langle 4n(J)\rangle \times \prod_{n(J)\ \mathrm{odd}} \boldsymbol{BO}\langle 4n(J)-2\rangle \times \prod \boldsymbol{K}(Z_2, \dim Z_i)$$

be given by $F = \prod f_J \times \prod f_i$.

COROLLARY 1.4. *F induces an isomorphism on cohomology* mod 2 *and hence an isomorphism* mod $\mathcal{C}_2$ *on homotopy groups, where $\mathcal{C}_2$ is the class of finite groups of odd order.*

We now give some of the geometric corollaries of the above theorems. If $[M] \in \Omega_n^{\mathrm{Spin}}$, $\pi^J(M) \in KO^{-n}(pt)$ denotes the characteristic number defined by π^J (see [2]).

COROLLARY 1.5. *Let $[M] \in \Omega_*^{\mathrm{Spin}}$. Then $[M] = 0$ if and only if $\pi^J(M) = 0$ for all J and all Stiefel-Whitney numbers of M vanish.*

COROLLARY 1.6. *Let $[M] \in \mathfrak{N}_*$. Then $[M]$ contains a* Spin *manifold if and only if all Stiefel-Whitney numbers of M involving W_1 and W_2 vanish.*

THEOREM 1.7. *Let* $\mathfrak{N}_* = \mathrm{Im}(\Omega_*^{\mathrm{Spin}} \to \mathfrak{N}_*)/\mathrm{Im}$ (*squares in* $\Omega_* \to \mathfrak{N}_*$). *Then* $\mathfrak{N}_n = 0$, $n \leqq 28$, $n \neq 24$. $\mathfrak{N}_{24} = Z_2 = \mathfrak{N}_{29}$, $\mathfrak{N}_{30} = \mathfrak{N}_{31} = 0$, $\mathfrak{N}_{32} = Z_2 + Z_2$, $\mathfrak{N}_{33} = Z_2$, $\mathfrak{N}_{34} = Z_2$, $\mathfrak{N}_{35} = 0$, $\mathfrak{N}_{36} = Z_2$, $\mathfrak{N}_{37} = Z_2 + Z_2$, $\mathfrak{N}_{38} = 0$, $\mathfrak{N}_{39} = Z_2 + Z_2$, $\mathfrak{N}_{40} = Z_2 + Z_2 + Z_2 + Z_2 + Z_2$, $\mathfrak{N}_{41} = Z_2 + Z_2 + Z_2$, $\mathfrak{N}_{42} = Z_2 + Z_2 + Z_2$, $\mathfrak{N}_{43} = Z_2 + Z_2$, $\mathfrak{N}_{44} = Z_2 + Z_2 + Z_2$. (*Compare* [**5**] *and* [**3**].)

The authors would like to thank Charles Sims who programmed the hard part of the above computation on an IBM 7094. These were done before we proved our main theorems and were helpful in checking our conjectures; in particular they led to the correct version of Theorem 1.3, and they gave us explicit Stiefel-Whitney numbers which detect elements in $\mathfrak{N}_n$, $n \leqq 44$.

Let $n(J)$ be even. Then there are elements $[M_J] \in \Omega_{4n(J)}^{\mathrm{Spin}}$ of infinite order which are detected by $\{P_J\}$ mod 2. If $J = (j_1, \cdots, j_k)$ has only even integers, we may take $M_J = QP^{j_1} \times \cdots \times QP^{j_k}$ where QP^n is the quaterionic projective n space. Let $n(J)$ be odd. Then there are elements $[N_J] \in \Omega_{4n(J)-2}^{\mathrm{Spin}}$ of order two which are detected by $\{Y_J\}$. Note $[N_J] \times \alpha = 0$, where $0 \neq [\overline{S}^1] = \alpha \in \Omega_1^{\mathrm{Spin}} = Z_2$ by Corollary 1.5. Let $\partial W_J = N_J \times \overline{S}^1$, and let $\partial V_2 = \overline{S}^1 \times S^0$. Then $\partial(W_J \times S^0) = \partial(N_J \times V_2)$. Let $M_J = W_J \times S^0 \cup N_J \times V_2$. Then $[M_J] \in \Omega_{4n(J)}^{\mathrm{Spin}}$ (the indeterminacy in this construction is all multiples of two). Let $\tau \in \Omega_4^{\mathrm{Spin}} = Z$ be a generator, and let $\omega \in \Omega_8^{\mathrm{Spin}} = Z \oplus Z$ be an element such that $\hat{A}(\omega) = 1$ (see [**6**]).

COROLLARY 1.8. *A basis for* $\Omega_*^{\mathrm{Spin}} \otimes Q$ *is given by* $[M_J] \times \omega^k$ *and* $[M_J] \times \tau \times \omega^k$, $k = 0, 1, \cdots$.

COROLLARY 1.9. $\mathrm{Ker}(\Omega_*^{\mathrm{Spin}} \to \Omega_*)$ *is a vector space over* Z_2 *with a basis given by* $[M_J] \times \omega^k \times \alpha^i$, $k = 0, 1, \cdots$, $i = 1, 2$, $n(J)$ *even and* $([M_J] \times \tau)/4 \times \omega^k \times \alpha^i$, $k = 0, 1, \cdots$, $i = 1, 2$, $n(J)$ *odd.*

Let $[M_i] \in \Omega_{\dim Z_i}^{\mathrm{Spin}}$ correspond to Z_i.

COROLLARY 1.10. *A basis for* $\Omega_*^{\mathrm{Spin}} \otimes Z_2$ *is given by*

(1) $[M_J] \times \omega^k \times \alpha^i$, $k = 0, 1, \cdots$, $i = 0, 1, 2$, $n(J)$ *even,*
(2) $[M_J] \times \tau \times \omega^k$, $k = 0, 1, \cdots$, $n(J)$ *even,*
(3) $[M_i]$,
(4) $[N_J]$, $n(J)$ *odd,*
(5) $[M_J] \times \omega^k$, $k = 0, 1, \cdots$, $n(J)$ *odd, and*
(6) $(([M_J] \times \tau)/4) \times \omega^k \times \alpha^i$, $k = 0, 1, \cdots$, $i = 0, 1, 2$, $n(J)$ *odd.*

The rank of Z in dimension n can be computed inductively from Theorem 1.3 and the information given in the following theorem.

THEOREM 1.11. (a) *The Poincaré polynomial for* $H^*(M\mathrm{Spin})$ *is*

$$\prod_{n>3;\, n\neq 2^r+1} (1 - t^n)^{-1}.$$

(b) *The Poincaré polynomial for* $\mathcal{A}$ *is*

$$\prod_{n=2^r-1;\, r\geq 1} (1 - t^n)^{-1}.$$

(c) *The Poincaré polynomial for* $\mathcal{A}/\mathcal{A}(Sq^1, Sq^2)$ *is*

$$\prod_{n=2^r-1;\, r\geq 3} (1 - t^n)^{-1}(1 - t^4)^{-1}(1 - t^6)^{-1}.$$

(d) *The Poincaré polynomial for* $\mathcal{A}/\mathcal{A}(Sq^3)$ *is*

$$\prod_{n=2^r-1;\, r\geq 3} (1 - t^n)^{-1}(1 - t^4)^{-1}(1 - t^6)^{-1}(1 + t + t^2 + t^3 + t^4).$$

We remark that Ω_*^{Spin} has no odd torsion (see [7]) and so the above theorems determine Ω_*^{Spin} as an additive group.

Finally, we comment that we can prove similar theorems for the complex spinor group, $\mathrm{Spin}^C = \mathrm{Spin} \times_{Z_2} U(1)$. In this case a manifold is cobordant to zero if and only if all its Stiefel-Whitney and Pontrjagin numbers are zero.

2. **Techniques of proofs.** Let $\mathcal{A}_1$ be the subalgebra of $\mathcal{A}$ generated by Sq^0, Sq^1, and Sq^2. Let $\mathcal{B}$ be a graded $\mathcal{A}_1$-module.[2] Note that $Q_0, Q_1 \in \mathcal{A}_1$ and that $Q_0Q_0 = 0$ and $Q_1Q_1 = 0$. We say that $\mathcal{B}$ has isomorphic homologies if $(\mathrm{Ker}\, Q_0 \cap \mathrm{Ker}\, Q_1)/(\mathrm{Im}\, Q_0 \cap \mathrm{Im}\, Q_1) \to H(\mathcal{B}, Q_i)$ is an isomorphism for $i = 0, 1$. The following theorem is a generalization of a theorem due to Wall [9] and seems to be of interest in itself.

THEOREM 2.1. *Let* $\mathcal{B}$ *be an* $\mathcal{A}_1$*-module with isomorphic homologies. Then, as an* $\mathcal{A}_1$*-module,* $\mathcal{B}$ *is isomorphic to a direct sum of four types of* $\mathcal{A}_1$*-modules, namely* $\mathcal{A}_1/\mathcal{A}_1(Sq^1, Sq^2) = Z_2$, $\mathcal{A}_1/\mathcal{A}_1(Q_0, Q_1)$, $\mathcal{A}_1/\mathcal{A}_1(Sq^3)$, *and* $\mathcal{A}_1$.

In order to prove Theorem 1.2 we note that $\mathcal{B} = H^*(BSO)$ satisfies Theorem 2.1, we use the knowledge of $KO^*(BSO)$ [1], and compute in the so-called Atiyah-Hirzebruch spectral sequence for $KO^*(BSO)$.

Theorem 1.3 follows from the following algebraic theorem.

THEOREM 2.2. *Let X and Y be graded vector spaces over Z_2. Let* $\mathcal{B}$ *be a connected coalgebra over* $\mathcal{A}$ *such that* $\mathrm{Ker}(\phi : \mathcal{A} \to \mathcal{B}) = \mathcal{A}(Sq^1, Sq^2)$, *where* $\phi(a) = a(1)$. *Given* $\theta' : (\mathcal{A}/\mathcal{A}(Sq^1, Sq^2) \otimes X) \oplus (\mathcal{A}/\mathcal{A}(Sq^3) \otimes Y) \to \mathcal{B}$ *such that* $\theta'_* : H((\mathcal{A}/\mathcal{A}(Sq^1, Sq^2) \otimes X) \oplus (\mathcal{A}/\mathcal{A}(Sq^3) \otimes Y), Q_i) \to H(\mathcal{B}, Q_i)$ *is*

[2] We assume that $\mathcal{B}_i = 0$ if $i < 0$ and that $\mathcal{B}_i$ is finitely generated.

an isomorphism for $i=0, 1$. *Define* $Z=\mathcal{B}/\mathrm{Im}\ \theta'+\bar{\mathcal{A}}\mathcal{B}$ *and extend* θ' *to* $\theta\colon (\mathcal{A}/\mathcal{A}(Sq^1, Sq^2)\otimes X)\oplus(\mathcal{A}/\mathcal{A}(Sq^3)\otimes Y)\oplus(\mathcal{A}\otimes Z)\to\mathcal{B}$ *in the obvious way. Then* θ *is an isomorphism.*

The proofs of the corollaries to Theorem 1.3 are not difficult if one applies the techniques developed in [2].

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