MANIFOLDS WITH $\pi_1 = Z$

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In this note we announce some results extending results of S. P. Novikov ([6] and [7]), the author [2], and C. T. C. Wall [8]. In the above papers it is shown how to characterize the homotopy type of 1-connected smooth closed manifolds of dimension $n \geqq 5$, $n \neq 2 \bmod 4$, and how to reduce the diffeomorphy classification of such manifolds to homotopy theory, with similar results for bounded manifolds in [8]. We show how to adapt these techniques to manifolds with $\pi_1 = Z$ and get analogous results. By studying the "mapping torus" using these results one may obtain results on existence and pseudo-isotopy of diffeomorphisms, (see [3]).

One has for example the situation of a closed smooth manifold M^n and a map $f: M^n \to X$, such that the normal bundle ν of M in S^{n+k} is induced by f from a bundle ξ over X. One does surgery on M with respect to the map f, i.e. if W is the cobordism determined by the surgery, then f extends to a map $F: W \to X$ such that the normal bundle of W in $S^{n+k} \times I$ is induced from ξ by F. In case M is simply connected many conditions facilitate the surgery, such as the Whitney embedding theorem, and the Hurewicz theorem, so that, with appropriate hypothesis on X and ξ, it is often possible to do surgery to create a manifold homotopy equivalent to X. The case of a nonzero fundamental group poses many problems, but if $\pi_1 M = Z$, one can reduce the situation to the simply connected case by using extra geometrical structure. The idea is to consider a 1-connected manifold U^n with two 1-connected boundary components, $\partial U = A_0 \cup A_1$, with $f: A_0 \to A_1$ a diffeomorphism, and consider the identification space M^n of U with $a \in A_0$ identified to $f(a) \in A_1$. Then M^n is closed and connected with $\pi_1 M = Z$, and it can be shown using surgery that any smooth connected M^n with $\pi_1 M = Z$, $n \geqq 5$ can be represented this way. One may then study U and A_0, A_1 using the techniques of surgery on 1-connected manifolds and then use this to obtain information about M.

In §1 we deal with closed manifolds and in §2, with manifolds with boundary. In §2 we examine in particular the case of homology circles, which gives certain results on the complements of higher dimensional knots (e.g. Corollaries 2.3 and 2.4).

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We will refer only to smooth manifolds, but we note that the analogous theorems may be proved for piecewise linear manifolds using the results and techniques of [**4**].

1. **Closed manifolds with $\pi_1 = Z$.** Let A and Y be 1-connected finite polyhedra and let $f: A\times\dot{I}\rightarrow Y$, where $\dot{I} =$ (the boundary of the interval I) $= \{0, 1\}$, so that f can be considered two maps, f_0 and f_1, of A into Y. Let $X = A\times I\cup_f Y$ i.e. the identification space of $A\times I\cup Y$ with $(a, t) = f(a, t)$ for $t\in\dot{I}$. We will say that "X is split by A." Note that $\pi_1 X = Z$, if X has a splitting. We will think of A as $A\times\frac{1}{2}\subset X$, and for convenience we will assume f an inclusion so that $A\times 0$ and $A\times 1\subset Y$.

If $X' = (A'\times I)\cup_{f'} Y'$, then a map h of splittings of X into X' is a pair of maps (g, k) such that

$$\begin{array}{ccc} A & \xrightarrow{g} & A' \\ f_i\downarrow & & \downarrow f_i' \\ Y & \xrightarrow{k} & Y' \end{array}$$

commutes. A map of splittings induces a map $\bar{h}$ of X into X'. The map (g, k) is an isomorphism of splittings if g and k are homotopy equivalences, in which case $\bar{h}$ is a homotopy equivalence.

DEFINITION. A connected finite polyhedron X satisfies Poincaré duality in dimension n if $H_n X = Z$ and $\cap g: H^i(X)\rightarrow H_{n-i}(X)$ is an isomorphism for all i, where g is a generator of $H_n(X)$. A splitting $X = (A\times I)\cup_f Y$ of a connected space X satisfies Poincaré duality in dimension $(n+1)$ if

(1) X satisfies Poincaré duality in dimension $n+1$,

(2) A satisfies Poincaré duality in dimension n,

(3) For $q\geqq n$, $f_{0*} = f_{1*}: H_q(A)\rightarrow H_q(Y)$ and is an isomorphism.

EXAMPLES. (1) A closed connected combinatorial $(n+1)$-manifold M, $n\geqq 4$, with $\pi_1 M = Z$ has a splitting satisfying Poincaré duality in dimension $n+1$.

(2) Let A be a 1-connected complex satisfying Poincaré duality in dimension n and let $f: A\rightarrow A$ be a homotopy equivalence of degree $+1$. Then the "mapping torus" of f defined by $X =$ the identification space of $A\times I$ with $(a, 0) = (fa, 1)$ has a splitting by A satisfying Poincaré duality in dimension $(n+1)$. Any fibre space over S^1 whose fibre F is 1-connected and satisfies Poincaré duality in dimension n, and such that $\pi_1(S^1)$ acts trivially on $H_n(F)$ is the homotopy type of such an X.

Let M^n be a smooth closed n-manifold, ν^k its normal k-plane bundle in S^{n+k}, for k large. By the Tubular Neighborhood Theorem, the total space $E(\nu)$ may be embedded in S^{n+k}, as a neighborhood of $M^n \subset S^{n+k}$, and thus we get a map $S^{n+k} \to T(\nu) = E(\nu)/\partial E(\nu)$ which is an isomorphism on H_{n+k}, and the homotopy class $C_M \in \pi_{n+k}(T\nu)$ of this map we will call a "normal invariant" of M. Note that the normal invariant is not uniquely defined; a bundle equivalence of ν with itself sends one normal invariant of M into another. A similar construction defines the normal invariant for a manifold with boundary $(M, \partial M)$ as a set of elements in $\pi_{n+k}(T(\nu), T(\nu | \partial M))$.

Suppose X is split by A and let ξ be a k-plane bundle over X. Then $E(\xi | (A \times (0, 1))) = E(\xi | A) \times (0, 1) = E((\xi | A) \oplus \epsilon^1)$ where ϵ^q is the trivial q-plane bundle. Set $(\xi | A) \oplus \epsilon^1 = \eta$ and we have that $E(\eta)$ is an open subset of $E(\xi)$ and there is a natural collapsing map of Thom complexes $q: T(\xi) \to T(\eta)$, since $T(\eta) = T(\xi)/T(\xi | Y)$. If X is an $(n+1)$-manifold, split by n-submanifold A, and if ξ is the normal bundle of X in S^{n+k+1}, then η is normal bundle of A in S^{n+k+1} and q_* sends the normal invariants of X into normal invariants of A.

If $H_n(Y) = Z$, $f_{i*}: H_n A \to H_n Y$ is an isomorphism for $i = 1, 2$ and $H_1(Y) = H_1(A) = 0$ for $i > n$, it follows using the Mayer-Vietoris sequence for $X = (A \times I) \cup_f Y$ that $H_{n+1}(X) = Z$, and that $q_*: H_{n+k+1}(T(\xi)) \to H_{n+k+1}(T(\eta))$ is an isomorphism (where both groups are Z, as seen from the Thom isomorphism).

THEOREM 1.1. *Let X be a space with a splitting satisfying Poincaré duality in dimension $(n+1)$, $n \geqq 4$, where the space A which splits X is a smooth n-manifold. Let ξ^k be an oriented k-plane bundle over X, k large, such that $\eta = (\xi | A) \oplus \epsilon^1$ is isomorphic to the normal bundle ν^{k+1} of A for some embedding of A in S^{n+k+1}. Let $\alpha \in \pi_{n+k+1}(T\xi)$ be such that $q_*(\alpha) \in \pi_{n+k+1}(T(\eta))$ is a normal invariant for A, where $q: T(\xi) \to T(\eta)$ is the collapsing map. Suppose that n is even or that $n = 4r - 1$ and $\langle L_r(\bar{p}_1, \cdots, \bar{p}_r), g \rangle = \sigma(X)$, where L_r is the Hirzebruch polynomial, $\sigma(X)$ is the signature of the quadratic form $H^{2r}(X; Q) \oplus H^{2r}(X; Q) \to H^{4r}(X; Q) = Q$, $\bar{p}_i$ are the dual Pontrjagin classes of ξ and $g \in H_{n+1}(X)$ is the preferred generator. Then there is a smooth compact manifold M^{n+1} split by A, and an isomorphism h of splittings from M to X, which is the identity on A. Further $h^*(\xi)$ is the normal bundle of $M^{n+1} \subset S^{n+k+1}$ and α corresponds to the normal invariant of M. If $n = 5$, 13 the same conclusion holds except for the normal invariant.*

(An extension to $n = 8q+1$ and ξ an SU*-bundle is possible using results of* [**1**]).

The general idea of the proof is as follows: Using a transverse regularity argument as in [**2**], we obtain a smooth closed manifold

$N^{n+1} \subset S^{n+k+1}$, with normal bundle γ, a map $f: N \to X$ of degree 1, and a bundle map $b: \gamma \to \xi$ covering f such that $T(b)_*(C_N) = \alpha \in \pi_{n+k+1}(T\xi)$. Using transversality again we may change f and b by homotopies so that $f^{-1}A =$ is a closed n-manifold $B \subset N$, $f|B$ is of degree $+1$, $(\gamma|B) \oplus \epsilon^1$ is the normal bundle δ to B in S^{n+k+1}, and b induces a bundle map b' of δ into ν such that $T(b')_*(C_B) = q_*(\alpha)$. Then there exists a cobordism W of B with A and a bundle map of the normal bundle of W into ν extending b'. Taking the union of $N \times I$ and $W \times I$, identifying the product neighborhood $B \times I \times 1 \subset \partial(N \times I)$ with $B \times I \subset W \times I$ and rounding corners, we obtain a cobordism of N to N', where $N' \supset A$, and with a bundle map of its normal bundle into ξ, etc. Cutting N' along A we get a manifold U, with $\partial U = A_0 \cup A_1$, A_i diffeomorphic to A, $i=0, 1$, $N' = U$ with A_0 identified to A_1 and a map $(U, A_0, A_1) \xrightarrow{h} (Y, A \times 0, A \times 1)$. Further h is covered by a bundle map of the normal bundle of U into $\xi | Y$ and h is a homotopy equivalence on A_i, $i=1, 2$. Then we proceed to do surgery on interior U till it is homotopy equivalent to Y, preserving the bundle map, and leaving everything the same on ∂U. Finally we identify A_0 and A_1 again to get the closed manifold M desired.

Let M, M_1, M_2 be closed oriented $(n+1)$-manifolds, $n \geqq 4$, split by closed 1-connected submanifolds A, A_1, A_2, respectively, and let ν, ν_1, ν_2 = normal k plane bundles of M, M_1, M_2 in S^{n+k+1}, k large. Let $g_i: (M_i, A_i) \to (M, A)$, $i=1, 2$ be homotopy equivalences induced by isomorphisms of splittings, such that $g_i^*(\nu) = \nu_i$, $i=1, 2$, where ν_i is the normal bundle of M_i and let $b_i: \nu_i \to \nu$ be a bundle equivalence covering g_i, $i=1, 2$. Suppose $f_i: S^{n+k+1} \to T(\nu_i)$ is a collapsing map for the embedding of M_i in S^{n+k+1} and $\eta: T(\nu) \to T(\nu | A \oplus \epsilon^1)$, and let $\alpha_i = T(b_i) f_i$, $i=1, 2$.

THEOREM 1.2. *If $h: S^{n+k+1} \times I \to T(\nu)$ is a homotopy between α_1 and α_2 such that ηh is homotopic* rel $S^{n+k+1} \times 0 \cup S^{n+k+1} \times 1$ *to the constant homotopy then M_1 is diffeomorphic to $M_2 \# \Sigma$, where $\Sigma \in \theta^{n+1}(\partial\pi)$* (i.e. *$\Sigma$ is a homotopy $(n+1)$-sphere which bounds a π-manifold*).

THEOREM 1.3. *If $n+1$ is odd >6 and α_1 is homotopic to α_2 then M_1 is diffeomorphic to $M_2 \# \Sigma$, for some $\Sigma^{n+1} \in \theta^{n+1}(\partial\pi)$.*

Suppose $\pi_1 M^{n+1} = Z$, M^{n+1} orientable $(n+1)$-manifold and let $\beta: S^1 \times D^n \subset M^{n+1}$ such that $\beta | S^1 \times 0$ represents a particular chosen generator of $\pi_1 M$. Let Σ^n be a homotopy sphere. Define $M^{n+1} \#_\beta \Sigma^n$ to be $(M - (S^1 \times D^n)) \cup (S^1 \times \Sigma^n - S^1 \times D^n)$ identified on their boundary by the obvious diffeomorphism of $S^1 \times S^{n-1}$. One can also think of this as twisting $S^1 \times D^n$ in M be using the diffeomorphism of S^{n-1}

corresponding to Σ^n, on each $x_0 \times S^{n-1} \subset S^1 \times D^n$ to attach $S^1 \times D^n$ to its complement, instead of the identity map. One can show that the result depends only on Σ, up to diffeomorphism, if $\Sigma \in \theta^n(\partial\pi)$.

THEOREM 1.4. *If $n+1$ is even and α_1 is homotopic to α_2 then M_1 is diffeomorphic to $M_2 \#_\beta \Sigma^n$, for some $\Sigma^n \in \theta^n(\partial\pi)$.*

2. **Manifolds with boundary; complements of certain knots.** Let (A, B) and (S, T) be polyhedral pairs with A, B, S, T 1-connected, let $f: (A, B) \times \dot{I} \to (S, T)$, and let (P, Q) be the pair defined by $P = (A \times I) \cup_f S$, $Q = (B \times I) \cup_{f'} T$, where $f' = f|B \times \dot{I}$. We say $(P, Q) = ((A, B) \times I) \cup_f (S, T)$ is split by (A, B). There are the analogous notions of maps and isomorphisms of splittings of pairs.

A pair (P, Q) is said to satisfy relative Poincaré duality in dimension $(n+1)$ if $\partial: H_{n+1}(P, Q) \to H_n(Q)$ is an isomorphism of infinite cyclic groups and in the commutative diagram

$$\begin{array}{ccccccc} \cdots \to & H^k(P, Q) & \longrightarrow & H^k(P) & \longrightarrow & H^k(Q) & \to \cdots \\ & \downarrow \cap g & & \downarrow \cap g & & \downarrow \cap(\partial g) & \\ \cdots \to & H_{n+1-k}(P) & \to & H_{n+1-k}(P, Q) & \to & H_{n-k}(Q) & \to \cdots \end{array}$$

all the vertical maps are isomorphisms, where g is a generator of $H_{n+1}(P, Q)$.

A splitting of (P, Q) is said to satisfy Poincaré duality in dimension $(n+1)$ if

(1) (P, Q) satisfies relative Poincaré duality in dimension $(n+1)$.

(2) (A, B) satisfies relative Poincaré duality in dimension n.

(3) For $q \geqq n$, $f_{0*} = f_{1*}: H_q(A, B) \to H_q(S, T)$, $f'_{0*} = f'_{1*}: H_{q-1}(B) \to H_{q-1}(T)$, and f_{i*}, f'_{i*} are isomorphisms.

EXAMPLES. (1) If W^{n+1} is oriented smooth or p.l., $i_*: \pi_1(\partial W) \cong \pi_1(W) = Z$, $n > 4$, then $(W, \partial W)$ has a splitting satisfying Poincaré duality in dimension $(n+1)$.

(2) Let (A, B) be a pair of 1-connected spaces satisfying relative Poincaré duality in dimension n, and let $f: (A, B) \to (A, B)$ be a homotopy equivalence of pairs of degree $+1$. Then the pair of mapping tori of f and $f|B$ has a splitting by (A, B) satisfying Poincaré duality in dimension $n+1$.

THEOREM 2.1. *Let (P, Q) be a pair with a splitting satisfying Poincaré duality in dimension $(n+1)$, $n \geqq 6$. Let ξ be an oriented k-plane bundle over P, k large, and let $\alpha \in \pi_{n+1+k}(T(\xi), T(\xi|Q))$ be such that $h(\alpha) = \Phi(g)$, where $g \in H_{n+1}(P, Q)$ is a generator, $\Phi: H_{n+1}(P, Q)$*

$\to H_{n+1+k}(T(\xi), T(\xi|Q))$ *is the Thom isomorphism. Then there is a unique smooth* $(n+1)$*-manifold with boundary* $(W, \partial W)$ *split by an* n*-submanifold* $(M, \partial M) \subset (W, \partial W)$ *with an isomorphism* h *of splittings* $(W, \partial W)$ *by* $(M, \partial M)$ *into the splitting of* (P, Q) *by* (A, B), *with* $\bar{h}^*\xi$ $=$ *normal bundle of* W *in* D^{n+k+1}, *and* α *corresponding to the normal invariant.*

This is the analog of Wall's theorem [8]. We make use of Wall's theorem first to make (A, B) a smooth manifold with boundary, and then the proof proceeds in the spirit of the proof of Theorem 1.1, adapting the ideas of Wall's proof [8] to this case where needed.

If one adds the hypothesis in (2.1) that (A, B) is a smooth manifold with boundary and that $q_*(\alpha) =$ a normal invariant for (A, B), one can prove the existence result with $n \geqq 5$, (cf. (1.1)).

Now we consider the special case of homology 1-spheres.

LEMMA 2.2. *Let* (X, Y) *be a pair with a splitting satisfying Poincaré duality in dimension* $(n+1)$, *and suppose* $H_*(X) \cong H_*(S^1)$. *Then for large* k, *any oriented bundle* ξ^k *over* X *is trivial, and* $\pi_{n+1+k}(T(\xi), T(\xi|Y))$ $= Z + Z_2$. *The Hurewicz homomorphism* h *sends the generator* α *of* Z *onto a generator* $\Phi(g)$ *of* $H_{n+k+1}(T(\xi), T(\xi|Y))$ *and there is a bundle equivalence of* ξ *which sends* α *into* $\alpha + \beta$, *where* β *generates* Z_2.

This is easy to prove using the natural map $(X, Y) \to (S^1 \times D^n, S^1 \times S^{n-1})$. Then (2.1) and (2.2) imply:

COROLLARY 2.3. *Let* (X, Y) *be a pair with a splitting by* (A, B) *satisfying Poincaré duality in dimension* $(n+1)$, $n \geqq 6$ *and suppose* X *is a homology* 1*-sphere. Then there is a manifold with boundary* $(W, \partial W)$ *split by* $(M, \partial M)$ *and an isomorphism of splittings of* $(W, \partial W)$ *by* $(M, \partial M)$ *into the splitting of* (X, Y) *by* (A, B). *Further such* $(W, \partial W)$ *is unique up to diffeomorphism and is a* π*-manifold. Finally, if* Y *is homotopy equivalent to* $S^1 \times S^{n-1}$ *then* W *is the complement of a smooth knot* (S^{n+1}, Σ^{n-1}), $(\Sigma^{n-1}$ *is a smooth homotopy* $(n-1)$*-sphere in* $S^{n+1})$.

COROLLARY 2.4. *Let* (S^{n+2}, Σ_i^n), $i = 1, 2$, $n \geqq 5$ *be smooth knots, let* $W_i =$ *the complement of an open tubular neighborhood of* Σ_i *in* S^{n+2}, *and suppose that all the homotopy groups* $\pi_k(W_i)$ *are finitely generated abelian groups for all* k, *for* $i = 1, 2$ (*so in particular* $\pi_1(W_i) = Z$). *Then* W_1 *is diffeomorphic to* W_2 *if and only if the pairs* $(W_1, \partial W_1)$ *and* $(W_2, \partial W_2)$ *are homotopy equivalent.*

One applies the Theorem of [5] to get a map of splittings with the right properties and applies Corollary 2.3.

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