THE GEOMETRY OF G-STRUCTURES[1]

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1. **Introduction.** Differential geometry studies differentiable manifolds and geometric objects or structures on them. It is now customary to distinguish it from differential topology by the presence of a structure in addition to the differentiable structure. What a differential geometric structure is or should be is a matter of taste. At the present state of the field a definition general enough to include all the significant structures will certainly contain many uninteresting ones. Among the attempts at a general definition is the notion of a geometric object initiated by Oswald Veblen [**130**].

Not all the geometrical structures are "equal". It would seem that the riemannian and complex structures, with their contacts with other fields of mathematics and with their richness in results, should occupy a central position in differential geometry. A unifying idea is the notion of a G-structure, which is the modern version of a local equivalence problem first emphasized and exploited in its various special cases by Elie Cartan [**36**], [**41**], [**129**]. Generally we will restrict ourselves in this article to the discussion of problems which fall under the notion of a G-structure.

Two general problems are of importance:

I. *Existence or nonexistence of certain structures on a manifold.*

EXAMPLE 1. A positive definite riemannian structure always exists.

EXAMPLE 2. On a compact manifold M a nowhere zero differentiable vector field exists if and only if the Euler-Poincaré characteristic of M is zero.

EXAMPLE 3. One may ask whether a nonzero vector field exists on M (supposed to be compact and orientable) which is parallel with respect to a riemannian metric. By Hodge's harmonic forms a necessary condition is that the first Betti number b^1 of M is $\geqq 1$. One can further prove that the second Betti number $b^2 \geqq b^1 - 1$ (cf. §5). These conditions are probably not sufficient.

II. *Local and global properties of a given structure.*

EXAMPLE 1. For a riemannian structure this means riemannian geometry.

EXAMPLE 2. If we are only interested in the existence or nonexist-

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ence of a nowhere zero vector field, it is not necessary to distinguish between contravariant and covariant vector fields, because a riemannian metric will transform one into the other. However, the distinction is essential when we study their properties. Any two nonzero contravariant vector fields are locally equivalent. On the other hand, a covariant vector field is the same as a linear differential form ω. To ω one can associate a pair of integers (k, l), defined to be the largest integers such that

$$(1) \qquad (d\omega)^k \neq 0, \qquad \omega \wedge (d\omega)^l \neq 0 \qquad l = k \text{ or } k - 1.$$

These vector fields define differential systems. Of importance is the study of the local and global properties of their integral manifolds.

It is natural to combine I and II and ask for conditions such that a structure exists with given local or global properties. A simple problem of this nature is: What are the conditions that there exists a covariant vector field on a manifold with given values of k, l $(=k$ or $k-1)$ at every point? I do not know the answer even in the following special case: Does there exist on a compact orientable three-dimensional manifold a linear differential form ω such that $\omega \wedge d\omega \neq 0$ everywhere? The tangent bundle of an orientable three-dimensional manifold is a product bundle, but this fact does not seem to help the problem.

Both in the last problem and in the problem of Example 3, I the conditions in question are differential conditions (i.e., involving the partial derivatives of the tensor fields in question), in contrast to Examples 1, 2, I, where the conditions are algebraic in nature. For problems of the latter kind various techniques for their treatment have been developed in fiber bundles (obstructions, characteristic classes, cohomology operations, etc.). For existence and properties of structures satisfying differential conditions much less is known in the way of general methods, except perhaps the conclusions which can be derived by application of harmonic forms. We will consider in this paper principally structures satisfying differential conditions.

When manifolds have certain structures, it is natural to consider their mappings which are in a sense admissible. Examples are isometric mappings of riemannian manifolds and holomorphic mappings of complex manifolds. Studies of such mappings and of problems mentioned above all lead to systems of differential equations or inequalities, generally nonlinear.

This paper will be devoted to a review of some of the important developments in differential geometry in recent years, following the above guideline. With a task of this scope omissions are inevitable and results unmentioned are in no way less important. We will emphasize simple and concrete problems, at the expense of generality.

2. **Riemannian structure.** A riemannian manifold of two dimensions has at every point a scalar invariant, the gaussian curvature. In higher dimensions its generalization is the riemannian or sectional curvature, which is a function of (p, λ), p being a point and λ a two-dimensional plane element through p. Geometrically this is the gaussian curvature at p of the surface through p generated by the geodesics through p and tangent to λ. The knowledge of the sectional curvature for all (p, λ) determines the Riemann-Christoffel tensor and hence in a sense all the local properties of the riemannian structure. The class of riemannian manifolds whose sectional curvature keeps a constant sign is of obvious geometrical interest.

A complete riemannian manifold with sectional curvature $\leqq 0$ has a universal covering space homeomorphic to the euclidean space. This is due to the fact that the geodesics through a point 0 contain no point conjugate to 0. By studying the isometries in the universal covering space one derives properties of the fundamental group of the manifold. An example is the following theorem [**112**]: A compact riemannian manifold with sectional curvature <0 has a fundamental group which cannot be abelian and of which every abelian subgroup is cyclic.

The Euler-Poincaré characteristic of a compact orientable riemannian manifold M of even dimension $n=2m$ is given by the Gauss-Bonnet formula [**4**], [**39**],

$$\chi(M) = \frac{(-1)^m}{2^{3m}\pi^m m!} \cdot \int_M \left(\sum_{i,j} \epsilon_{i_1\cdots i_{2m}} \epsilon_{j_1\cdots j_{2m}} R_{i_1 i_2 j_1 j_2} \cdots R_{i_{2m-1} i_{2m} j_{2m-1} j_{2m}} \right) dV \tag{2}$$

where dV is the volume element and R_{ijkl} are the components of the curvature tensor relative to orthonormal frames; $\epsilon_{i_1\cdots i_{2m}}$ is zero if $i_1, \cdots, i_{2m}$ do not form a permutation of $1, \cdots, 2m$, and is equal to 1 or -1 according as the permutation is even or odd. It does not seem to be an easy problem to draw conclusions on the sign of the integrand from properties of the sign of sectional curvature. As a consequence the following conjecture has not been decided in its full generality: If M has sectional curvature $\geqq 0$, then $\chi(M) \geqq 0$; if M has sectional curvature $\leqq 0$, then $\chi(M) \geqq 0$ or $\leqq 0$, according as $n \equiv 0$ or 2, mod 4. The statement is true for $n=4$ [**42**] and for the case that M has constant sectional curvature.

Much work has been done recently on complete riemannian manifolds with sectional curvature bounded below by a positive constant. Such a manifold is always compact. This is proved by examining the second variation of arc length and estimating the distance from a point to its first conjugate point on a geodesic. For $n=2$ the manifold is homeomorphic to the sphere. This is not true for higher dimensions. In fact, the complex projective space with the usual elliptic hermitian metric has sectional curvatures R satisfying the inequalities $A/4 \leqq R \leqq A$, A being a positive constant.

Again by considering the second variation of arc length Myers proved the following theorem [**98**]: A complete riemannian manifold with Ricci curvature $\geqq c$ (=constant)>0 is compact. (The Ricci curvature is a function of (p, X), where X is a vector through p. It is the arithmetic mean of the sectional curvatures at (p, λ_i), $1 \leqq i \leqq n-1$, where λ_i are $n-1$ mutually perpendicular plane elements through X.) It follows that its universal covering manifold is also compact and hence that its fundamental group is finite. From the last result one concludes that the first Betti number of the manifold is zero.

The last conclusion can also be derived by a method of Bochner [**141**]. Bochner's method is an extremely important tool in differential geometry. It applies to the high-dimensional Betti numbers of riemannian manifolds and, more importantly, to the derivation of criteria (in terms of curvature properties) for the vanishing of more general cohomology groups (cf. §8). The idea is to study the measure of a harmonic form of degree r and the curvature properties at a point where the measure attains its maximum. From the absence of any point with such curvature properties one concludes that the harmonic form must be zero and hence that the rth Betti number is zero. For $r=1$ one gets the above corollary to Myers' theorem. For $r>1$ the geometrical significance of Bochner's conditions has not been sufficiently studied (see below).

With the example of the complex projective space in mind Rauch introduced the notion of a pinched riemannian manifold. A riemannian is said to be α-pinched if its sectional curvatures R satisfy the inequalities: $\alpha A \leqq R \leqq A$ for a positive constant A. Rauch proved the following theorem: *A complete simply connected α-pinched riemannian manifold with $\alpha=0.75$ is homeomorphic to the sphere*, [**114**], [**115**].

By utilizing Bochner's conditions for the vanishing of the second Betti number Berger derived the following theorem [**17**]: *A compact α-pinched riemannian manifold M such that* dim $M=2m$ (*resp.* dim $M=2m+1$) *and* $\alpha>\frac{1}{4}$ (*resp.* $>2m-2/8m-5$) *has its second Betti number equal to zero.*

It was Klingenberg who observed that in the even-dimensional case a lemma of Synge can be utilized to give a simpler proof of Rauch's theorem and improve the result. Synge's lemma says the following: *Let M be a simply connected compact even-dimensional riemannian manifold whose sectional curvature is* >0. *Let g be a closed geodesic of minimum length among the closed geodesics of M. Then there is a family of closed curves converging toward g, which are all shorter than g.*

The combined efforts of Klingenberg and Berger lead to the following theorem [**15**], [**72**]: *Let M be a simply connected riemannian manifold which is* α*-pinched. If* $\alpha > \frac{1}{4}$, *M is homeomorphic to a sphere. If* $\alpha = \frac{1}{4}$ *and M is even-dimensional and not homeomorphic to a sphere, then M is a compact symmetric riemannian manifold of rank* 1 *with its canonical metric.* The proof of this theorem is difficult; the main tools are the Alexandrow-Rauch-Toponogov comparison theorem and the Morse critical point theory [**74**].

The symmetric riemannian manifolds of rank 1 mentioned above are the complex or quaternionic projective spaces or the Cayley plane. According to a theorem of H. C. Wang [**132**], these, together with the spherical and elliptic spaces, are the only compact and connected two-point homogeneous spaces.

Generally it is difficult to decide whether a differentiable manifold can be given a riemannian metric of positive curvature. The answer is not known even for the manifold $S^2 \times S^2$ (S^2 = two-dimensional sphere) and for Milnor's exotic spheres. Besides the examples mentioned above the only known simply connected spaces having a riemannian metric of positive curvature are two examples given by Berger [**16**]: These are homogeneous spaces of dimensions 7 and 13 respectively and are not homeomorphic to the sphere. No topological property is known for a compact simply connected manifold to carry a riemannian metric of positive curvature.

The homeomorphism theorem of Klingenberg and Berger was improved by D. Gromoll [**57**] and E. Calabi to a diffeomorphism theorem: *There exists a number* δ_n *depending on n* ($0 < \delta_n < 1$, $\lim_{n\to\infty} \delta_n = 1$) *such that if a simply connected riemannian manifold M of dimension n is* δ_n*-pinched, then M is diffeomorphic to the n-sphere with its usual differentiable structure. The best value of* δ_n *is not known; current estimates of it tend rapidly to* 1.

These pinching theorems have been extended to complex Kähler manifolds (cf. §5, 7). For instance, Andreotti and Frankel proved that if a compact Kähler manifold of complex dimension 2 has positive sectional curvature, it is complex analytically equivalent to the complex projective plane [52]. More general results have been obtained by doCarmo, S. Kobayashi, Klingenberg, Bishop-Goldberg, etc. [21], [22], [48], [73], [75].

Given a manifold, it is natural to ask for the "simplest" riemannian metric which can be defined on it. For a compact two-dimensional manifold this will be one of constant gaussian curvature; the curvature is positive for the sphere, zero for the torus, and negative for a surface of genus >1. For high dimensions constancy of the sectional curvature will lead to the space forms and it is known that not every manifold has a riemannian metric of constant curvature. A more general class of riemannian metrics consists of the Einstein metrics, which are defined by the condition that the Ricci tensor is a scalar multiple of the fundamental tensor. It is not known whether every simply connected manifold can have an Einstein metric. As an Einstein metric on a three-dimensional manifold is necessarily of constant curvature, the problem has a bearing on the Poincaré hypothesis that a compact simply connected three-dimensional manifold is homeomorphic to the sphere.

A result in this direction is the following theorem of Yamabe [140]: *A compact riemannian manifold of dimension* $\geqq 3$ *is conformally equivalent to one of constant scalar curvature. It is not known whether the sign of the scalar curvature is a global conformal invariant.*

3. **Connections** [5], [8], [40], [43], [50], [63], [78], [94]. The classical example of a connection is the parallelism of Levi-Civita in riemannian geometry. When the riemannian manifold is a surface in ordinary euclidean space, the parallelism along a curve is obtained by taking the tangent developable surface and rolling it on a plane. Parallelism is at the basis of the notion of curvature, for geometrically curvature measures the dependence of parallelism on the curve. For higher dimensions algebraic concepts enter and there is no extra work to develop the theory in a more general setting.

Let G be a Lie group of dimension r, and let $L(G)$ be its Lie algebra. On a differentiable manifold M we consider exterior differential forms (to be abbreviated as e.d.f.), with values in $L(G)$. A multiplication can be introduced as follows: Let X_i, $1 \leqq i \leqq r$, be a basis in $L(G)$. An $L(G)$-valued e.d.f. can be written

$$\lambda = \sum \lambda^i \otimes X_i, \tag{3}$$

where λ^i are ordinary e.d.f.'s. If

$$\mu = \sum_j \mu^j \otimes X_j \tag{4}$$

is a second $L(G)$-valued form, we define the product

$$(5)\qquad [\lambda, \mu] = \sum_{i,j} \lambda^i \wedge \mu^j \otimes [X_i, X_j].$$

This definition is clearly independent of the choice of basis in $L(G)$. If λ, μ are of degrees l, m respectively, we have

$$(6)\qquad [\lambda, \mu] = (-1)^{lm+1}[\mu, \lambda].$$

Let ω^i, $1 \leqq i \leqq r$, be the dual basis of X_i. By left translations they can be identified with left-invariant linear differential forms on the group manifold G. Then

$$(7)\qquad \omega = \sum_i \omega^i \otimes X_i$$

is an $L(G)$-valued left-invariant linear differential form on G, called the Maurer-Cartan form. The Maurer-Cartan equation can be written

$$(8)\qquad d\omega = -\tfrac{1}{2}[\omega, \omega].$$

Consider an inner automorphism of G defined by $s \to asa^{-1}$, where a, $s \in G$ and a is fixed. This leaves fixed the unit element of G and induces an automorphism on $L(G)$, which we call $\operatorname{ad}(a)$. Clearly $\operatorname{ad}(a)$ has the properties

$$(9)\qquad \begin{aligned} \operatorname{ad}(ab) &= \operatorname{ad}(a)\operatorname{ad}(b), \qquad a, b \in G \\ \operatorname{ad}(a)[\lambda, \mu] &= [\operatorname{ad}(a)\lambda, \operatorname{ad}(a)\mu], \end{aligned}$$

where λ, μ are the forms considered above.

A connection is most conveniently defined as a structure on a principal fiber bundle. A simple example of a principal fiber bundle is the space of all orthonormal frames in euclidean space. It is a fiber space over the euclidean space itself, the projection being defined by taking for an orthonormal frame its origin, so that a fiber consists of all orthonormal frames with the same origin. The generalization of this classical concept in geometry and kinematics leads to the method of moving frames of Elie Cartan in differential geometry and to the notion of a principal fiber bundle in modern algebraic topology. Formally the latter is defined by a differentiable mapping $\psi: B \to M$ and, relative to an open covering $\{U, V, \cdots\}$ of M, a family of "transition functions" $g_{UV}(x) \in G$, $x \in U \cap V$, defined for every pair of members of the covering satisfying $U \cap V \neq \phi$, such that: (1) $\psi^{-1}(U)$ is a product $U \times G$ and has the local coordinates (x, s), $x \in U$, $s \in G$; (2) For $x \in U \cap V$, the local coordinates (x, s), (x, t), s, $t \in G$, relative to U and V respectively, define the same point if and only if $s=g_{UV}(x)t$, the multiplication at the right-hand side being in the sense of group multiplication in G. Thus a fiber $\psi^{-1}(x)$ has the structure of a group manifold defined up to left translations. Right translations have a meaning, not only on a fiber, but on B itself; it is locally defined by $b=(x, s)\to ba=(x, sa)$.

At a point $b\in B$ we call the tangent space $V(b)$ to the fiber the vertical space. A subspace in the tangent space of B at b, which has only the zero vector in common with $V(b)$ and which, together with $V(b)$, spans the tangent space, is called a horizontal space. A connection in the bundle B is a field of horizontal spaces which is stable under the right translations of B. Instead of the horizontal spaces we can equally well define the connection by their orthogonal spaces in the cotangent spaces. This in turn can be described as an $L(G)$-valued linear differential form $\phi(b)$ in B, such that its restriction to a fiber is the Maurer-Cartan form and such that under right translations of B it satisfies the condition

$$\phi(ba) = \mathrm{ad}(a^{-1})\phi(b). \tag{10}$$

In terms of the local coordinates (x, s) in $\psi^{-1}(U)$, this condition means that $\phi(b)$ is of the form

$$\phi(b) = \omega(s) + \mathrm{ad}(s^{-1})\theta_U(x, dx), \tag{11}$$

where $\theta_U(x, dx)$ is an $L(G)$-valued linear differential form in U. By equating the expressions for $\phi(b)$ in $\psi^{-1}(U\cap V)$, we get

$$\omega(g_{UV}(x)) + \mathrm{ad}(\overset{-1}{g_{UV}})\theta_U(x, dx) = \theta_V(x, dx). \tag{12}$$

This formula gives the relation between θ_U, θ_V in $U\cap V$.

A sectionally smooth curve in B is called a horizontal curve, if it is everywhere tangent to a horizontal space of the connection. To a given point $b\in B$ all the elements $a\in G$ such that b and ba can be joined by a horizontal curve form a group H_b, called the holonomy group at b. It is easy to see that the holonomy groups H_b, $H_{b'}$ at two points b, $b'\in B$ are conjugate to each other in G.

The connection defines an absolute differentiation or covariant differentiation in B, as follows: Let η be an exterior differential form of degree q in B, with values in a vector space E. Its absolute differential $D\eta$ will be of degree $q+1$, also with values in E. To define it we consider an exterior differential form to be an alternating multilinear function of vector fields. If X is a vector field in B, we denote by VX and HX its projections in the vertical and horizontal spaces respectively, so that $X=VX+HX$. Then $D\eta$ is defined by the equation

(13) $$D\eta(X_1, \cdots, X_{q+1}) = d\eta(HX_1, \cdots, HX_{q+1}),$$

where $X_1, \cdots, X_{q+1}$ are $q+1$ vector fields in B.

Using this definition we compute the absolute differential of the connection form ϕ. By (11) we find

(14) $$d\phi + \tfrac{1}{2}[\phi, \phi] = \mathrm{ad}(s^{-1})\{d\theta_U + \tfrac{1}{2}[\theta_U, \theta_U]\}.$$

From this it is seen that the common expression is $D\phi$. Putting $\Phi = D\phi$ and

(15) $$\Theta_U = d\theta_U + \tfrac{1}{2}[\theta_U, \theta_U],$$

we have

(16) $$\Phi = \mathrm{ad}(s^{-1})\Theta_U.$$

The $L(G)$-valued quadratic differential form Φ is called the curvature form. In $U \cap V$ we have

(17) $$\Theta_U = \mathrm{ad}(g_{UV})\Theta_V.$$

In the usual treatment the curvature form is described by the family of forms $\{\Theta_U\}$ with the transformation law (17).

Now let G be a subgroup of a Lie group G'. Every element $a \in G$ defines an automorphism $\mathrm{ad}(a): L(G') \to L(G')$, which leaves $L(G)$ invariant. It induces a linear mapping of the quotient space $L(G')/L(G)$ into itself, which we also denote by $\mathrm{ad}(a)$. The situation occurs that we have a G-connection in a G'-bundle, or, more precisely, that we have a connection in the associated principal bundle with the group G', when the bundle B is considered to have the structural group G. With the covering $\{U, V, \cdots\}$ of M the curvature form of the connection will be given by a family of $L(G')$-valued quadratic differential forms $\{\Theta'_U\}$, satisfying the relation $\Theta'_U = \mathrm{ad}(g_{UV}(x))\Theta'_V$ in $U \cap V$. The projections of these forms into the quotient space $L(G')/L(G)$ constitute the torsion form.

The curvature forms define the local properties of a connection and the holonomy groups H_b its global property. Ambrose and Singer showed how the Lie algebra of H_b can be determined from the curvature forms [**5**].

Example. Consider the frame bundle of a manifold M of dimension n (a frame is an ordered set of n linearly independent tangent vectors with the same origin). In this case $G = \mathrm{GL}(n, R)$ and G can be considered to be the group of all $n \times n$ nonsingular matrices. Its Lie algebra has as underlying vector space that of all $n \times n$ matrices, nonsingular or singular. For $g \in G$, the Maurer-Cartan form is $\omega = g^{-1}dg$, while $\mathrm{ad}(g)X = gXg^{-1}$, $X \in L(G)$, in the sense of matrix multiplication. If u^i, v^k, $1 \leqq i, k \leqq n$, are local coordinates in U, V respectively, we have

$$g_{UV} = \left(\frac{\partial u^i}{\partial v^k}\right), \qquad \theta_U = \left(\sum_j \Gamma^i_{kj}\, du^j\right). \tag{18}$$

The transformation formula (12) becomes

$$\frac{\partial^2 u^i}{\partial v^k \partial v^j} = \sum_{m,l} \Gamma^i_{ml} \frac{\partial u^m}{\partial v^k}\frac{\partial u^l}{\partial v^j} + \sum_l \frac{\partial u^i}{\partial v^l} \tilde{\Gamma}^l_{kj} \tag{19}$$

where $\tilde{\Gamma}^l_{kj}$ are the coefficients in θ_V. This is the transformation formula for the "components" of an affine connection in the usual form.

Returning to the general case, it follows from (12) and a simple extension argument that a connection always exists in a principal fiber bundle. In the case of the frame bundle of a manifold this implies that the bundle space is topologically parallelisable, i.e., there are n^2+n linear differential forms in the space, which are everywhere linearly independent.

The notion of a connection is at the basis of many fields of differential geometry, such as riemannian, hermitian, as well as projective and conformal differential geometries. Our treatment adapts well both to the local theory and to the study of the homology (with real coefficients) of principal fiber bundles. A close relationship exists between the curvature forms and the characteristic classes of a fiber bundle (cf. §9).

4. *G*-**structure** [18], [33], [41], [129]. Among the important structures on a manifold is the reduction of the structural group of the tangent bundle, which we explain as follows:

Let T be an n-dimensional real vector space and T^* be its dual space. Denote their pairing by $\langle y, \xi\rangle \in R$, $y \in T$, $\xi \in T^*$. We let $\mathrm{GL}(n, R)$ act on T on the left and on T^* on the right, so that the following relation holds:

$$\langle gy, \xi\rangle = \langle y, \xi g\rangle, \qquad g \in \mathrm{GL}(n, R). \tag{20}$$

The tangent bundle over M has the local charts (x, y_U), $x \in U$, $y_U \in T$, which are the local coordinates of the tangent vectors relative to U. The local coordinates (x, y_U) and (x, y_V) in $U \cap V$ define the same tangent vector if and only if $y_U = g_{UV}(x) y_V$, where $g_{UV}\colon U \cap V \to \mathrm{GL}(n, R)$. Consider a subgroup G of $\mathrm{GL}(n, R)$; we say that the structural group of the tangent bundle is reduced to G, if all $g_{UV}(x) \in G$. Such a reduction will be simply called a G-structure.

EXAMPLE 1. An $O(n)$-structure is nothing else but a riemannian structure.

EXAMPLE 2. For $n=2m$ consider $\mathrm{GL}(m, C)$ as a subgroup of $\mathrm{GL}(n, R)$. Then a $\mathrm{GL}(m, C)$-structure is what is called an almost complex structure.

Many methods are known in the theory of fiber bundles to find necessary conditions for a manifold to have a G-structure (obstruction theory, characteristic classes, cohomology operations, etc.) and we will not discuss them. We will go further to study the properties of a given G-structure. We take a basis in T and call all frames obtained from this basis frame by the transformations of G permissible frames. If M has a G-structure, it makes sense to call a frame of M permissible, if it is so in a local coordinate system. The dual basis of a permissible frame in the cotangent space will be called a permissible coframe. In the associated principal bundle of the G-structure the permissible coframes give rise to n linearly independent linear differential forms, which are globally defined.

The problem of local invariants of a G-structure is essentially the following problem of equivalence of Elie Cartan [**33**]: Given a set of n linearly independent linear differential forms $\theta^i_U(u, du)$ in the coordinates u^k, and another such set $\theta^i_V(v, dv)$ in the coordinates v^k, $1 \leqq i, k \leqq n$, and given a Lie group $G \subset \mathrm{GL}(n, R)$. To determine the condition that there exist functions

$$v^i = v^i(u^1, \cdots, u^n), \tag{21}$$

such that the $\theta^i_V(v(u), dv(u))$ differ from the $\theta^i_U(u, du)$ by a transformation of G. The last condition gives rise to an exterior differential system. As a first step to the solution of the problem one considers the differential forms $\phi^i = \sum_j g^i_j \theta^j_U$, $(g^i_j) \in G$, with the coordinates of G as auxiliary variables. In our terminology these are the permissible coframes. We illustrate this by some examples for $n=2$:

EXAMPLE 3. $G=\mathrm{SO}(2)$ and is the group of all matrices of the form

$$\begin{pmatrix} \cos\lambda & -\sin\lambda \\ \sin\lambda & \cos\lambda \end{pmatrix}.$$

With λ as auxiliary variable we have

$$\phi^1 = \cos\lambda\theta^1 + \sin\lambda\theta^2, \qquad \phi^2 = -\sin\lambda\theta^1 + \cos\lambda\theta^2. \tag{22}$$

There is a linear differential form

$$\pi = d\lambda + \text{lin comb of } \theta^1, \theta^2, \tag{23}$$

which is uniquely determined by the conditions

(24) $$d\phi^1 = \pi \wedge \phi^2, \qquad d\phi^2 = -\pi \wedge \phi^1.$$

The exterior derivative of π is of the form

(25) $$d\pi = -K\phi^1 \wedge \phi^2.$$

One sees that K is the gaussian curvature of the riemannian structure. The forms ϕ^1, ϕ^2, π in the associated principal bundle with the group NSO(2) (we denote by NG the nonhomogeneous linear group whose homogeneous part is G) can be interpreted geometrically as defining a connection in the bundle.

EXAMPLE 4. G is the group of all matrices of the form

$$\begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix},$$

$\lambda \neq 0$. We put

(26) $$\phi^1 = \lambda\theta^1, \qquad \phi^2 = \lambda\theta^2.$$

There is a uniquely determined linear differential form

(27) $$\pi = \frac{d\lambda}{\lambda} + \text{lin comb of } \theta^1, \theta^2,$$

satisfying the conditions

(28) $$d\phi^1 = \pi \wedge \phi^1, \qquad d\phi^2 = \pi \wedge \phi^2.$$

We find

(29) $$d\pi = A\phi^1 \wedge \phi^2,$$

where A is of the form $A(u^1, u^2)/\lambda^2$. The forms ϕ^1, ϕ^2, π define a connection in the associated principal bundle with the group NG.

The structure can be interpreted as a three-web of plane curves in the sense of Blaschke [**24**], the curves being defined respectively by the equations

(30) $$\theta^1 = 0, \qquad \theta^2 = 0, \qquad \theta^1 + \theta^2 = 0.$$

The condition $A = 0$ is a necessary and sufficient condition that the curves can be locally mapped into three families of parallel straight lines. It can also be interpreted geometrically as the condition for a certain hexagonal configuration to exist.

EXAMPLE 5. G is the group of all matrices of the form

$$\begin{pmatrix} \lambda & -\mu \\ \mu & \lambda \end{pmatrix},$$

$\lambda^2+\mu^2\neq 0$. Again put

$$
(31)\qquad \begin{aligned}\phi^1 &= \lambda\theta^1 + \mu\theta^2,\\ \phi^2 &= -\mu\theta^1 + \lambda\theta^2.\end{aligned}
$$

There are linear differential forms π_1, π_2 satisfying the equations

$$
(32)\qquad \begin{aligned}d\phi^1 &= \pi_1 \wedge \phi^1 + \pi_2 \wedge \phi^2,\\ d\phi^2 &= -\pi_2 \wedge \phi^1 + \pi_1 \wedge \phi^2.\end{aligned}
$$

But they are not completely determined by these conditions. The structure is given by $(\theta^1)^2+(\theta^2)^2$, determined up to a positive factor. It is therefore the two-dimensional conformal structure. No connection can be introduced, which will depend on this structure alone; for otherwise it would mean that the local homeomorphisms leaving the conformal structure invariant would depend on a finite number of constants. This is a first example of a pseudo-group structure.

The general problem of equivalence is a problem on exterior differential systems of a certain type. In the real analytic case the theorem of Cartan-Kuranishi [**86**] says that in a finite number of steps the system can be prolonged either to a system without solution or to a system in involution. They correspond respectively to the cases that the two G-structures are locally inequivalent or equivalent. In the latter case there may be a pseudo-group of local homeomorphisms, other than the identity, which leaves the structures invariant. To carry out the general program of Cartan-Kuranishi in a particular case is not always an easy problem and very frequently leads to complicated calculations. We mention two simple instances where the problem of equivalence is not solved: (1) Almost complex structure $G=\mathrm{GL}(m, C)$, $n=2m$. The problem of equivalence was solved by Libermann for $m=2$ [**93**]. (2) Symplectic structure, where G is the linear group in a space of dimension $2m$ consisting of all transformations leaving invariant an exterior quadratic form of maximum rank. This is in other words the local classification of antisymmetric covariant tensor fields $a_{ij}=-a_{ji}$ of order two and maximum rank.

Two classical cases where the problem of equivalence is solved and a connection attached to the structure are the following:

EXAMPLE 6. $G=O(n)$. This is the riemannian structure where the Levi-Civita connection can be attached.

EXAMPLE 7. G is the group of all matrices of order n ($\geqq 3$) of the form λA, $\lambda>0$, where A is orthogonal. This is the conformal structure, to which a normal conformal connection, uniquely determined, can be attached.

There are various advantages to be gained in attaching a connection to a G-structure, if this is possible. First the structure is put in a general setting, whose results will then be applicable. Secondly, it will be possible to introduce geometrical concepts. In fact, the geometry of a connection along a curve is the same, whether the curvature form is zero or not (Fermi's theorem). Thus the geometry appears more clearly as a generalization of classical geometry. Thirdly, G-structures equivalent to a connection are in a sense simpler. For instance, the group of automorphisms leaving the structure invariant will be a Lie group.

It appears from the examples that it will be desirable to consider the associated principal bundle with the group NG. We are actually considering a bundle with the group NG with a cross-section into the associated bundle of homogeneous spaces NG$/G$, making it a bundle with the structural group G. In such a bundle there is a torsion form associated to a connection with the structural group NG. The question arises as to whether it is possible to attach uniquely a connection by properties of this torsion form. It can be proved that such a connection exists whenever G is compact [**43**]. In the particular case of the riemannian structure the Levi-Civita connection is completely determined by the vanishing of the torsion form. For general compact G there may be different ways of attaching the connection which will be characterized by different properties of the torsion form. In this context it would be a problem of some interest to determine the properties of the noncompact G, for which the answer to the above question is affirmative. (Cf. §13 on the Weyl-Cartan theorem.)

An almost complex structure whose torsion form (this can be defined, although there is no connection) is zero is called integrable. Newlander and Nirenberg proved that such an almost complex structure is subordinate to a complex structure [**101**], [**103**]. It follows that an almost complex manifold whose torsion form vanishes identically is a complex manifold. This is an example of a pseudo-group structure. No connection can be attached to a complex manifold from its complex structure alone.

Among the G-structures those to which a connection can be attached and those which are subordinate to a pseudo-group structure seem to be the extreme cases. They are among the most important G-structures. The others are probably too complicated to be of much geometrical interest.

A further generalization is to structures which involve elements of contact of higher order. An example is the projective geometry of paths of Veblen and T. Y. Thomas. Much work on the foundations of this generalization has been done by Ehresmann and his school.

5. **Harmonic forms** [64], [68], [119], [134]. A general method which gives global implications on the existence of geometrical structures satisfying differential conditions is the theory of harmonic differential forms of Hodge. Let M be a compact oriented n-dimensional riemannian manifold. Let Δ be the Laplace-de Rham operator on M. A differential form η is called harmonic, if $\Delta\eta = 0$. Since M is compact, every harmonic form is closed. All harmonic forms of degree q constitute a real vector space H^q. The fundamental theorem on harmonic forms says that the linear mapping $\rho: H^q \to H^q(M, R)$ ($=q$-dimensional cohomology space of M with real coefficients) defined by sending a harmonic form to its cohomology class in the sense of de Rham's theorem is a one-one isomorphism. We are going to consider the existence of certain structures on M, which will allow a finer analysis of the cohomology ring of M with real coefficients through the application of harmonic forms.

Let G be a subgroup of $\mathrm{SO}(n)$. Then a G-structure on M defines an orientation of M and a riemannian structure, to be called the associated riemannian structure. Such a G-structure is called holonomic, if it satisfies one of the following three conditions, which are equivalent [43], [84], [135]:

(1) The homogeneous holonomy group of the Levi-Civita connection (of the associated riemannian structure) is G or a subgroup of G.

(2) Under the Levi-Civita parallelism permissible frames remain permissible.

(3) There is a connection with the structural group NG, whose torsion form is zero.

It seems to us that these are the structures to which the harmonic forms can be applied most advantageously to derive global implications.

Particular cases of a holonomic G-structure include the following:

EXAMPLE 1. $G = U(m)$, $m = n/2$. This is essentially the Kähler structure on a complex manifold in its real formulation.

EXAMPLE 2. $G = \mathrm{SO}(p) \times \mathrm{SO}(n-p)$, $1 \leqq p \leqq n-1$. This means that the manifold has a field of p-dimensional plane elements parallel with respect to the Levi-Civita connection.

A G-structure on M allows a finer classification of its exterior differential forms. (This applies also to other tensor fields.) In fact, G acts on T^* and has an induced representation on $\Lambda^q(T^*)$, $0 \leqq q \leqq n$. Let W be an invariant subspace of $\Lambda^q(T^*)$ under the action of G. A q-form on M is said to be of type W if the element it associates to every point $x \in M$ belongs to the corresponding subspace W_x. If $G \subset O(n)$, then every invariant subspace W has an orthogonal space $W^\perp$ which is also invariant and we can define to a q-form η its orthogonal projection $P_W\eta$ into a q-form of type W. More generally, if Q: $W \rightarrow \Lambda^r(T^*)$ is a linear mapping which commutes with the action of G, we define an operator Q on q-forms of type W, the image being then an r-form. An example of such a mapping Q, besides P_W, is the exterior multiplication by an exterior $(r-q)$-form which is invariant under G. When there are such operators, it is of importance to find the conditions under which they commute with the Laplace-de Rham operator, and we have the theorem:

Let M have a holonomic G-structure. Then $P_W\Delta = \Delta P_W$. Moreover, if Q is the multiplication by an invariant exterior differential form, we have also $Q\Delta = \Delta Q$.

The proof of this theorem makes use of an explicit formula of R. Weitzenböck for Δ. (The formula allows a simple derivation of Bochner's theorem on the relation between curvature and Betti numbers of a compact riemannian manifold. Considering the importance of the operator Δ, it should have further applications.) It follows from the theorem that if η is harmonic, then $P_W\eta$ is also harmonic. Hence if $\Lambda^q(T^*)$ is a direct sum of the invariant subspaces $W_1, \cdots, W_k$, the space H^q of harmonic forms of degree q is a direct sum of the spaces of harmonic forms of types $W_1, \cdots, W_k$ respectively. The second part of the theorem leads to isomorphisms of subspaces of harmonic forms of different degrees. The analysis of the cohomology groups (over the real field) of M is reduced to a purely algebraic problem (namely, that of studying the induced representation of G on $\Lambda(T^*)$). From this one derives various global implications from the existence of a holonomic G-structure.

EXAMPLE 1 (CONTINUED). The group G leaves invariant an exterior form of degree 2 and maximum rank. Denote by Ω the corresponding differential form on M. Introduce on the exterior differential forms η of M the operators

$$L\eta = \Omega \wedge \eta, \qquad \Lambda = *^{-1}L*. \tag{33}$$

A form η is called primitive, if $\Lambda\eta = 0$. It follows from our commutativity theorem that every harmonic form η can be written uniquely as

$$\eta = \sum_{r \geq \max(0, m-p)} L^r \eta_r, \qquad p = \deg \eta \tag{34}$$

where η_r are harmonic and primitive (Hodge's decomposition theorem). For compact orientable manifolds with such a holonomic G-structure we can derive the following global properties:

(1) Every odd-dimensional Betti number is even.

(2) Let u be the cohomology class determined by Ω. The homomorphism $H^{r-2}(M, R) \to H^r(M, R)$ defined by $\alpha \to \alpha \cup u$ (=cup product of α, u), $\alpha \in H^{r-2}(M, R)$, is, for $r \leqq m$, an isomorphism.

(3) The homomorphism $H^r(M, R) \to H^{2m-r}(M, R)$, $r \leqq m$, defined by $\alpha \to \alpha \cup u^{m-r}$, $\alpha \in H^r(M, R)$, is a one-one isomorphism.

It is worth noting that these properties can be derived without reference to the complex structure.

EXAMPLE 2 (CONTINUED). In this case we can derive that the p-dimensional Betti number is $\geqq 1$. For $p=1$, a further analysis of the situation gives the inequality $b^2 \geqq b^1 - 1$, b^1 and b^2 being respectively the one and two-dimensional Betti numbers. It would be natural to ask whether for $p=1$ the conditions are now sufficient.

The use of harmonic forms also gives information on the multiplicative structure of the cohomology ring $H^*(M, R)$, particularly the index $\tau(M)$. The latter is defined as follows: $\tau(M)=0$, if dim $M \not\equiv 0$, mod 4. If dim $M=4k$, we consider in the real vector space $H^{2k}(M, R)$ the function $f(u, v)=(u \cup v)M$, $u, v \in H^{2k}(M, R)$, which is the value of the cohomology class $u \cup v$ on the fundamental class M. This function is a symmetric nondegenerate bilinear form. $\tau(M)$ is defined to be the number of its positive eigenvalues minus the number of its negative eigenvalues.

Consider now the space $\Lambda^{2k}(T^*)=W_1 \oplus W_2$, where W_1 (resp. W_2) is the subspace of all elements invariant under $*$ (resp. transformed to its negative by $*$). Let h_i be the dimension of the space of harmonic forms of degree $2k$ and type W_i, $i=1, 2$. Then a study of the multiplicative properties of harmonic forms gives the theorem: $\tau(M) = h_1 - h_2$. From this theorem Hodge's index theorem on compact Kähler manifolds can be derived by purely algebraic considerations.

We will mention another application of the above index theorem: Suppose that a compact orientable manifold of dimension $4k$ has $2k$ linearly independent vector fields which are parallel with respect to a riemannian metric. Then its index is zero. Examples (due to Atiyah, private communication) exist of four-dimensional compact manifolds with nonzero index but with two linearly independent vector fields (nonparallel with respect to any riemannian metric).

6. **Leaved structure** [51], [60], [116], [117]. From the point of view of §4 a leaved structure is a reduction of the structural group $G(n, R)$ of the tangent bundle of a manifold of dimension n to the subgroup

$$\begin{pmatrix} G(p, R) & 0 \\ * & * \end{pmatrix}, \qquad 1 \leqq p \leqq n-1,$$

such that a local differential condition is satisfied. More precisely, if the structural group of the tangent bundle is so reduced, that of the cotangent bundle will be reduced accordingly, and a field of permissible coframes will be given locally by the linear differential forms $(\theta^1, \cdots, \theta^n)$, which are linearly independent and are such that $\theta^{p+1}, \cdots, \theta^n$ are determined up to a transformation of $GL(n-p, R)$. A leaved structure is such a reduction satisfying the further conditions

$$d\theta^i \wedge \theta^{p+1} \wedge \cdots \wedge \theta^n = 0, \qquad p+1 \leqq i \leqq n. \tag{35}$$

By Frobenius' theorem this means that the pfaffian system

$$\theta^{p+1} = \cdots = \theta^n = 0 \tag{36}$$

is completely integrable, i.e., there exists at each point a local coordinate system $(x^1, \cdots, x^n)$ relative to which the system becomes

$$dx^{p+1} = \cdots = dx^n = 0. \tag{37}$$

The integer p is called the dimension of the leaved structure, and $n-p$ its codimension. A leaf is a "maximal" integral submanifold.

The simplest example of a leaved structure is given by a differential equation of the first order in the plane. A consequence of the Poincaré-Bendixon theory says that the leaves go to infinity in both directions. The problem of classifying the leaved structures in the plane was solved by W. Kaplan [**69**]. In particular, Kaplan proved that to a leaved structure in the plane there exists a continuous real-valued function in the plane, which has neither a maximum nor a minimum and which is constant on the leaves. In this respect it may be of interest to mention that Wazewski gave examples of C^∞-leaved structures in the plane such that any differentiable function in the plane which is constant on the leaves is a constant.

Another example is the leaved structures (of dimension 1) on a torus [**125**]. Two extreme cases are: (1) All leaves are closed. (2) The leaves are ergodic. If the torus is taken to be a unit square with opposite sides identified, the second case occurs when the directions of the field makes a constant angle $\alpha\pi$ (α irrational) with the sides. The fundamental theorem of Denjoy says that (under sufficient smoothness hypotheses) if there is no closed leaf, then every leaf is ergodic. Denjoy's work has been extended by A. J. Schwartz to arbitrary closed surfaces [**120**]. If we interpret a leaved structure of dimension 1 as the action of a surface by the real line, we define a minimal set of this action to be a nonempty closed invariant set, which contains no proper subset with the same property. The Denjoy-Schwartz theorem says that if a compact connected two-dimensional surface is under the action of the real line, then a minimal set is: (1) either a single point; (2) or a closed curve; (3) or the whole surface, the last case happening only for the torus.

Another simple example of a leaved structure is given by a coset decomposition of a Lie group relative to a subgroup.

A notable example of a leaved structure of codimension one on the three-sphere S^3 was given by G. Reeb [**116**]. Geometrically this can be described as follows: Consider S^3 as the union of two solid tori with a two-dimensional torus as common boundary. In each solid torus take a leaved structure of codimension one whose integral submanifolds are like paraboloids with the boundary torus as a limiting surface. These leaved structures can be fitted together to give a leaved structure on S^3. The same construction cannot be extended to a sphere of higher odd dimension, and it is an open problem whether S^{2m+1}, $m>1$, has a leaved structure of codimension one.

Reeb's leaved structure is not analytic and he raised the question whether an analytic one (of codimension one) exists on S^3. The answer was given as negative by A. Haefliger, who proved the following theorem [**59**]: *Let M be a compact real analytic manifold which has a real analytic leaved structure of codimension one. Then the fundamental group $\pi_1(M)$ is not finite.*

If a manifold has a leaved structure of dimension p, it must have a G-structure, with $G=\mathrm{GL}(p, R)$. The problem, called fundamental by Reeb, is whether this condition is sufficient. In fact, it is not known whether any $\mathrm{GL}(p, R)$-structure is homotopic to a leaved structure of dimension p, i.e., whether they can be connected by a differentiable family of $\mathrm{GL}(p, R)$-structures. A compact orientable three-dimensional manifold is parallelisable, but it is an open question whether it has a leaved structure of codimension one.

When a manifold has a leaved structure, an important invariant is the holonomy group in the sense of Ehresmann. It gives an accurate description of the behavior of the leaves in the neighborhood of a leaf. An important problem is whether a leaved structure has a compact leaf. On the three-sphere S^3 the existence of a compact leaf for a leaved structure of dimension one (respectively two) was conjectured by H. Seifert (respectively by H. Kneser) [**121**]. A proof of Kneser's conjecture has been announced by S. P. Novikov [**106**].

Let $X_1, \cdots, X_p$ be vector fields, which span the tangent spaces of the leaves. Then condition (35) is equivalent to the condition that the brackets $[X_i, X_j]$ are linear combinations of $X_1, \cdots, X_p$. Milnor defined as the rank of a manifold the maximum number of vector fields $X_1, \cdots, X_p$, everywhere linearly independent, such that $[X_i, X_j]=0$, $1 \leqq i, j \leqq p$. Lima proved that S^3 has rank one [**95**]. This was generalized by Lima and H. Rosenberg to the result that the rank of a compact simply connected manifold of dimension n is at most $n-2$. The problem is closely related to the study of the action of noncompact transformation groups on a manifold. Except when the transformation group is one-dimensional (flows on a manifold), very little is known.

An important notion on the leaved structure is that of "structural stability," which was initiated by Andronov and Pontrjagin [**7**] in the case of a disk and whose study has so far been restricted to one-dimensional leaves. A structural stable vector field is one such that the topological properties of the trajectories remain invariant under a small perturbation of the vector field (relative to some topology). Peixoto proved that the structurally stable vector fields on a compact two-dimensional surface are open and dense in the set of all vector fields on the surface [**109**]. The study in the higher-dimensional case is wide open. In fact, Smale gave examples of manifolds in which the structurally stable vector fields are not dense in the set of all vector fields. What makes the theory in higher dimensions much more difficult is the presence of recurrence. An important recent result is the "closing lemma" of C. C. Pugh [**113**]. Among its consequences is the theorem that the closure of the set of all nonwandering points is the closure of the set of all closed trajectories.

7. **Complex structure** [**66**], [**134**]. The existence of a complex structure on a manifold is a nontrivial fact, so that an understanding of complex manifolds should begin with some examples. Two obvious necessary conditions are even dimensionality and orientability. An orientable manifold of two (real) dimensions always has a complex structure. The difficulty in proving this is the local theorem of Korn and Lichtenstein to the effect that a two-dimensional (positive definite) C^∞-riemannian manifold is locally conformal to the euclidean plane. For higher dimensions the simplest examples are the m-dimensional complex euclidean space $E_m(C)$ (or simply E_m) and the m-dimensional complex projective space $P_m(C)$ (or P_m). (E_1 is known as the gaussian plane and P_1 as the Riemann sphere in complex function theory.) The next examples are to be their quotient manifolds and submanifolds. Compact submanifolds of P_m are, by a theorem of Chow, the same as the nonsingular algebraic varieties. Of course there are no compact submanifolds in E_m, other than points, as there is no nonconstant analytic function on a compact complex manifold.

An example of a quotient manifold of E_m is E_m/Δ, where Δ is the discrete group generated by $2m$ translations linear independent over the reals. This is called a complex torus. Depending on the choice of Δ, there are some (those satisfying the Riemann conditions) which are isomorphic to algebraic varieties and there are also some on which there exists no nonconstant meromorphic function. The former are called abelian varieties.

It is equally possible to take an open submanifold of E_m and consider a quotient space of it. If $z_1, \cdots, z_m$ are the coordinates of E_m and $0=(0, \cdots, 0)$, then the discrete group Δ_1 generated by the transformations

$$z_k' = 2z_k, \qquad 1 \leqq k \leqq m, \tag{38}$$

has no fixed point in E_m-0 and $(E_m-0)/\Delta_1$ is a complex manifold. This is called a Hopf manifold. Topologically it is homeomorphic to $S^1\times S^{2m-1}$. For $m>1$ a Hopf manifold cannot be given a Kähler structure and is therefore not algebraic, because the latter must have its second Betti number $\geqq 1$.

To find further complex manifolds an obvious process is to form cartesian products. More significant are the blowing-up process (σ-process of H. Hopf) and fiber space constructions.

An example of blowing-up is as follows: In the projective plane P_2 let a line element (p, L) be the pair consisting of a point p and a line L through p. Consider all line elements (p, L) such that L passes through a given point b of P_2. They form a complex manifold M_2 of two (complex) dimensions. There is a complex analytic mapping $f\colon M_2\to P_2$, defined by $f(p, L)=p$. Then f is one-to-one for $M_2-f^{-1}(b)$ and $f^{-1}(b)$ is isomorphic to P_1. We describe this geometrically by saying that M_2 is obtained from P_2 by blowing up a point. Generally we can blow up a point b in any complex manifold M_m, leaving the points of M_m-b unchanged and replacing b by P_{m-1}. The same process generalizes to the blowing-up of any nonsingular submanifold. The process has its origin in the birational transformation of algebraic varieties, where it is known as a quadratic transformation. It exists only for $m>1$. Blanchard proved that if a compact Kähler manifold is blown up along a nonsingular submanifold, the resulting manifold is again Kählerian [23]. The process is of great importance in the theory of complex manifolds.

A simple application is to a construction due to Andreotti. Let T be a complex torus group of dimension $m>1$. (The difference between a complex torus and a complex torus group is that the latter has the additive group structure, i.e., a certain point singled out as the neutral element of the group.) Let α be the automorphism $x\to -x$, $x\in T$, and consider the quotient space T/α. The latter has singularities, 2^{2m} in number, arising from the fixed points of α, which are of a very simple type. They can be resolved by the blowing-up process, leading to a nonsingular complex manifold K, called the Kummer manifold. K is simply connected (Spanier [127]). If $m=2$ and T is an abelian variety, K is the classical Kummer surface. If T has no nonconstant meromorphic function, the same is true of K. Thus we get an example of a simply-connected compact Kähler manifold which has no meromorphic functions other than constants.

If we are interested in compact complex manifolds, the simplest class of fiber spaces consists of those of complex tori. In fact, a Hopf manifold is a fiber bundle of one-dimensional complex tori over a complex projective space, with transition functions which are holomorphic. In order to have a sufficiently wide class of complex manifolds, it is desirable to relax the condition of a fiber space, in the sense that the fibers are not required to be isomorphic to each other as complex manifolds. Following Blanchard, Calabi, Atiyah, Bott, a class of fiber spaces of complex tori can be constructed as follows [9]: In E_{2m} let G be the Grassmann manifold of all m-dimensional vector spaces through the origin. Let A be the real vector space of dimension $2m$ imbedded as a subset of E_{2m}. Consider the subset $J\subset G$ consisting of those m-dimensional vector spaces which have only the zero vector in common with A; J is an open subset of G. Over J there is the universal vector bundle of dimension m, attaching to each point of J the corresponding vector space. There is also the trivial vector bundle of dimension $2m$ over J. Let E' be their quotient bundle. The basis vectors of E_{2m} give by projection in each fiber of E' $2m$ vectors which are linearly independent over the reals. The quotient space of E' by these $2m$ vector fields is then a fiber space of complex tori over J. One can construct a compact complex manifold out of it by considering its restriction to a compact submanifold of J. Such a compact submanifold is, for instance, the manifold of all the m-dimensional vector spaces through the origin and lying on the quadric cone

$$z_1^2+\cdots+z_{2m}^2=0,$$

where $z_1,\cdots,z_{2m}$ are the coordinates in E_{2m}. It can be seen that this is the homogeneous space $O(2m)/U(m)$. By a theorem of Blanchard [23] the compact complex manifold so obtained (the fiber space!) which, as a differentiable manifold, is the product $O(2m)/U(m)\times T^{2m}$ (T^{2m} is a real torus of real dimension $2m$), has no Kählerian structure. On the other hand, $O(2m)/U(m)\times T^{2m}$ has a Kählerian complex structure derived from those of the two factors. We get in this way a manifold which has both Kählerian and non-Kählerian complex structures. The problem is unsolved whether there exists a simply-connected compact manifold with the same property.

Among the nonsingular algebraic varieties a notable family consists of the ruled surfaces, and in fact ruled surfaces with a directrix line. In P_{n+3} with the homogeneous coordinates $(x_0, x_1, \cdots, x_{n+3})$ consider the normal curve

$$x_0 = x_1 = 0, \qquad x_2 = 1, \qquad x_3 = t, \cdots, x_{n+3} = t^{n+1}$$

in the linear subspace of dimension $n+1$ with the equation $x_0 = x_1 = 0$. The lines joining the point of the normal curve with the parameter t to the point $(1, t, 0, \cdots, 0)$ generates a ruled surface without singularities. We denote it by Σ_n. As a real manifold it is a bundle of S^2 over S^2. Topologically there are only two classes of bundles of S^2 over S^2: the cartesian product $S^2 \times S^2$ and another. It can be shown that Σ_n is differentiably the product $S^2 \times S^2$ if and only if n is even. As complex manifolds the Σ_n are distinct from each other (a theorem of Hirzebruch [**65**]). Thus $S^2 \times S^2$ is a simply connected manifold which admits an infinite number of complex structures.

Another class of complex manifolds arises from a different context, namely those which admit a transitive group of complex analytic automorphisms, the so-called homogeneous manifolds. If G is a connected compact Lie group and T a maximal toroid of G, Borel observed that G/T has a complex structure [**25**]. Goto, Borel, and Weil proved that it is an algebraic variety and Goto proved that it is rational [**56**], [**124**]. Wang determined all the compact homogeneous complex manifolds with a finite fundamental group, among which are the simply connected even-dimensional compact Lie groups [**133**]. Many of these manifolds are non-Kählerian, and are thus far from being algebraic. The problem requires a detailed analysis of Lie algebras.

The question of the complex structure on a manifold can be generally divided into four stages:

(a) Existence of an almost complex structure. This is mainly a question on fiber bundles.

(b) Existence of a complex structure on an almost complex manifold. If a manifold is known to have an almost complex structure, then the methods of (a) do not give further information. The integrability conditions will easily decide whether the given almost complex structure is subordinate to a complex structure. If it is not, no general method is known to find out whether there is a complex structure (except of course when one is given). In this respect the question whether S^6 has a complex structure remains one of the most urgent problems on complex manifolds. A remarkable recent result of van de Ven, not yet published, gives an example of a compact four-dimensional almost complex manifold which has no complex structure. The absence of a complex structure on this manifold follows from the Riemann-Roch-Hirzebruch formula for arbitrary compact complex manifolds, which in turn is a consequence of the Atiyah-Singer index theorem (see §10).

(c) Determination of all complex structures on a manifold. Riemann's mapping theorem says that S^2 has a uniquely determined complex structure. Hirzebruch's example of $S^2 \times S^2$ has an infinite number of complex structures. It is not known whether $S^2 \times S^2$ has other complex structures than those given by Hirzebruch. Andreotti proved that there are no other algebraic structures [**6**]. Andreotti also proved that there is only one Kähler structure on P_2 (the standard one); Hirzebruch and Kodaira proved that there is only one Kähler structure on P_m for odd m [**67**].

(d) If a manifold has a continuum of complex structures, it is important to give a structure to this continuum in a natural way. The classical moduli problem on compact Riemann surfaces consists in making an analytic space of dimension $3g-3$ out of the set of complex structures on a Riemann surface of genus $g>1$ [**19**]. In high dimensions Frölicher and Nijenhuis proved that the complex structure on a compact complex manifold M is rigid, if $H^1(M, \Theta)=0$, where $H^1(M, \Theta)$ is the one-dimensional cohomology group of M over the sheaf of germs of holomorphic vector fields (cf. §8) [**53**]. Subsequently Kodaira and Spencer made extensive studies of the deformation of complex structures [**83**]. The crowning achievement is the theorem of M. Kuranishi, which says that, for any compact complex manifold, there exists a universal family of deformations [**87**], [**88**].

8. **Sheaves** [**31**], [**38**], [**54**], [**58**], [**123**]. The theory of sheaves is one of the most basic and natural concepts on manifolds with a structure. Let us begin by its definition:

Let M be a topological space. A sheaf of abelian groups on M, or simply a sheaf, is given by: (1) A function $x \rightarrow S_x$, which associates to every point $x \in M$ an abelian group S_x (to be written additively); (2) A topology (not necessarily satisfying separation axioms) in $S = \bigcup_{x \in M} S_x$, such that the following conditions are satisfied (ψ denoting the mapping $S \rightarrow M$, defined by $\psi(S_x)=x$):

(S_1) The mapping $\sigma \rightarrow -\sigma$ which, to each $\sigma \in S$ associates the inverse $-\sigma$ of σ in the group $S_{\psi(\sigma)}$, is a continuous mapping of S into itself. The mapping $(\sigma, \tau)\to\sigma+\tau$ defined for the set $R=\{(\sigma, \tau) \mid \psi(\sigma)=\psi(\tau)\}$ is a continuous mapping of R into S.

(S_2) ψ is a local homeomorphism, i.e., every $\sigma\in S$ has a neighborhood V such that the restriction of ψ to V is a homeomorphism of V onto an open subset of M.

EXAMPLE 1. THE CONSTANT SHEAF. Let G be an abelian group, and let $S=M\times G$.

The following notions are defined in an obvious way: subsheaf, quotient sheaf, homomorphism of sheaves, exact sequence of sheaves. Also we can define in the same way sheaves of other algebraic structures, such as rings, ideals, modules and even nonabelian groups.

In practice a sheaf is given by a presheaf, which is a family of abelian groups S_U associated to the members of an open covering $\{U, V, \cdots\}$ of M, such that the following conditions are satisfied: (1) To every pair (U, V) with $V\subset U$ there is a homomorphism $f_{VU}\colon S_U\to S_V$; (2) To every triple (U, V, W) with $W\subset V\subset U$ we have $f_{WU}=f_{WV}\circ f_{VU}$. To $x\in M$ we define S_x to be the inductive limit of S_U with $U\ni x$, etc.

EXAMPLE 2. Let S_U be the additive group of continuous real-valued functions in U. To $V\subset U$ define f_{VU} to be the restriction. In this way we get the sheaf of germs of continuous real-valued functions. Similarly, there is the sheaf of germs of differentiable functions on a differentiable manifold, holomorphic functions on a complex manifold, etc.

Let M be a topological space and S a sheaf of abelian groups over M. For every $q\geq 0$ we can define by the standard Čech theory a cohomology group $H^q(M, S)$, S being called the coefficient sheaf. To an open set $U\subset M$ denote by $\Gamma(U, S)$ the group of all sections over U (a section over U is a continuous mapping $s\colon U\to S$ such that $\psi\circ s=\text{identity}$). Let $\{U_i\}$ be an open covering of M. Then a cochain of dimension q is an element $f_{i_0\ldots i_q}\in\Gamma(U_{i_0}\cap\cdots\cap U_{i_q}, S)$, which is alternating in its indices and is zero if $U_{i_0}\cap\cdots\cap U_{i_q}=\emptyset$. The coboundary is defined in a standard way. The cohomology group of the covering is the quotient group of the group of all cocycles of dimension q over the subgroup of coboundaries. The inductive limit of the cohomology groups of the coverings is the cohomology group $H^q(M, S)$.

A homomorphism of sheaves $S\to S'$ induces a homomorphism $H^q(M, S)\to H^q(M, S')$ of the corresponding cohomology groups. From now on assume M to be paracompact. To a subsheaf R of S we can define the natural homomorphism

$$\delta^q: H^q(M, S/R) \to H^{q+1}(M, R).$$

Then the fundamental property of cohomology is the following: Let

$$0 \to R \to S \to T \to 0$$

be an exact sequence of sheaves on M. The sequence of cohomology groups

$$0 \to H^0(M, R) \to H^0(M, S) \to H^0(M, T) \to H^1(M, R) \to H^1(M, S) \to \cdots$$

is exact. (The homomorphisms are those defined above.)

EXAMPLE 3. Meaning of $H^0(M, S)$. Let $\{U_i\}$ be a covering of M. A 0-cochain is formed by the sections s_i: $U_i \to S$. It is a cocycle if and only if $s_i = s_j$ in $U_i \cap U_j \neq \emptyset$. Hence $H^0(M, S)$ is the group of all global sections of S.

EXAMPLE 4. Meaning of $H^1(M, S)$. A one-cochain is given by the sections s_{ij}: $U_i \cap U_j \to S$, whenever $U_i \cap U_j \neq \emptyset$. It is a cocycle if and only if $s_{ij} + s_{jk} + s_{ki} = 0$ for $U_i \cap U_j \cap U_k \neq \emptyset$. If S is the sheaf of germs of mappings into a topological group Γ (Γ not necessarily abelian), $H^1(M, S)$ is the set of all equivalence classes (in the sense of bundles) of bundles with the group Γ.

EXAMPLE 5. DE RHAM'S THEOREM. *Let M be a compact differentiable manifold. Let A^p (resp. C^p) be the sheaf of germs of exterior differential forms (resp. closed ext. diff. forms) of degree p. In particular, C^0 is the constant sheaf of real numbers. Then the sequence*

$$0 \to C^p \xrightarrow{i} A^p \xrightarrow{d} C^{p+1} \to 0,$$

where i is the injection and d is the exterior differentiation, is exact. (The essential point in the proof of this exactness is the so-called Poincaré lemma, which says that any closed differential form is locally an exterior derivative.) From this follows the exactness of the corresponding sequence of cohomology groups. But A^p is a fine sheaf and $H^q(M, A^p) = 0$, $q \geqq 1$. (A fine sheaf means essentially that its global sections can be localized.) It follows from the exact sequence that

$$H^q(M, C^p) \cong H^{q+1}(M, C^{p-1}), \qquad q > 0.$$

By applying this isomorphism successively, we get

$$H^{p+q}(M, R) \cong H^0(M, C^{p+q})/dH^0(M, A^{p+q-1})$$

which is precisely the statement of de Rham's theorem.

EXAMPLE 6. It is in complex manifolds (a "richer" structure) that the sheaf theory is most useful. The simplest example is Dolbeault's theorem. Let M be a compact complex manifold. Let $A^{p,q}$ (resp.

$B^{p,q}$) be the sheaf of germs of exterior differential forms (resp. d''-closed) of type (p, q). Then the sequence of sheaves

$$0 \longrightarrow B^{p,q} \xrightarrow{i} A^{p,q} \xrightarrow{d''} B^{p,q+1} \longrightarrow 0$$

is exact. (The proof that d'' is onto is here more difficult.) The sheaf $A^{p,q}$ is fine, while the sheaf $B^{p,0}$ is the sheaf of germs of holomorphic differential forms of degree p. As in Example 3, we derive from the exact sequence of cohomology groups the isomorphism

$$H^r(M, B^{p,0}) \cong H^0(M, B^{p,r})/d''H^0(M, A^{p,r-1})$$

which is the statement of Dolbeault's theorem.

EXAMPLE 7. COUSIN'S PROBLEM. In a complex manifold M let S be the sheaf of germs of meromorphic functions and Ω the subsheaf of germs of holomorphic functions. A section of the quotient sheaf S/Ω is a system of principal parts. The classical additive Cousin problem consists in deciding whether such a system of principal parts is that of a global meromorphic function. It follows from the exact sequence of cohomology groups that this Cousin problem always has a solution if $H^1(M, \Omega) = 0$.

EXAMPLE 8. Kodaira-Spencer's classification of complex analytic line bundles over a compact Kähler manifold [82]. The structural group is the multiplicative group C^* of nonzero complex numbers. If Ω^* denotes the sheaf of germs of nonzero holomorphic functions, the group of complex line bundles over a complex manifold M is $H^1(M, \Omega^*)$. To determine this group we consider the exact sequence of sheaves

$$0 \longrightarrow Z \xrightarrow{j} \Omega \xrightarrow{e} \Omega^* \longrightarrow 0,$$

where j is the injection and e is defined by $e(f(z)) = \exp(2\pi i f(z))$. If M is compact, we get the exact sequence

$$0 \rightarrow H^1(M, Z) \xrightarrow{j} H^1(M, \Omega) \xrightarrow{e} H^1(M, \Omega^*) \xrightarrow{\delta} H^2(M, Z) \rightarrow \cdots .$$

The homomorphism δ associates to every complex line bundle its characteristic class. The subgroup of all complex line bundles with characteristic class zero is therefore isomorphic to $H^1(M, \Omega)/jH^1(M, Z)$. If M is a compact Kähler manifold, this can be proved to be a complex torus, which is the Picard variety of M.

This example involves the second cohomology group.

In Examples 6, 7, 8 we have seen some applications of sheaf theory to complex manifolds. More significant applications arise from the notion of an analytic coherent sheaf. Let M be a complex analytic manifold. Let $\lambda: \Omega \rightarrow M$ be the sheaf of germs of holomorphic functions. A sheaf $\psi: S \rightarrow M$ is called analytic, if, for every $x \in M$, S_x has a module structure over the ring Ω_x, such that the mapping $(f, \alpha) \rightarrow f\alpha$ defined for the subset $R = \{(f, \alpha) \mid \lambda(f) = \psi(\alpha)\}$, is a continuous mapping of R into S.

EXAMPLE 9. Let Ω^q be q copies of the sheaf Ω; an element of Ω_x^q is an ordered set $(f_1, \cdots, f_q)$ of q germs of holomorphic functions at x. Let Ω_x act on Ω_x^q according to the formula

$$f(f_1, \cdots, f_q) = (ff_1, \cdots, ff_q).$$

Then Ω^q is an analytic sheaf.

Let S and S' be two analytic sheaves on M. A homomorphism $f: S \rightarrow S'$ of sheaves is called analytic if, for every $x \in M$, the homomorphism $f_x: S_x \rightarrow S'_x$ is compatible with the operations of Ω_x. The kernel, the image and the cokernel of f are then all analytic sheaves.

An analytic sheaf S over M is called coherent if every $x \in M$ has an open neighborhood U such that the induced analytic sheaf $S(U)$ is isomorphic to the cokernel of an analytic homomorphism $f: \Omega^q(U) \rightarrow \Omega^r(U)$ (q and r integers). In particular, an analytic sheaf is called locally free if every $x \in M$ has an open neighborhood U such that the induced sheaf $S(U)$ is isomorphic to $\Omega^q(U)$ for a certain integer q.

Stein manifolds. The Stein manifolds (of dimension >0) are noncompact complex manifolds which generalize the domains of holomorphy and which possess a sufficiently large number of holomorphic functions. Precisely speaking, a Stein manifold is a complex manifold with a countable base, satisfying the following conditions:

(1) To any two points $x, y \in M$, $x \neq y$, there exists a holomorphic function f in M, such that $f(x) \neq f(y)$.

(2) To every point $x \in M$ there exist holomorphic functions in M, which form a local coordinate system at x.

(3) M is holomorphically convex.

The following fundamental theorem (due to Oka, Cartan, and Serre) accounts for most of the properties of Stein manifolds:

A complex manifold M is a Stein manifold if and only if the following two properties hold:

(A) *For any coherent sheaf S over M the module of global sections of S over M generates at every point $x \in M$ the module of local sections.*

(B) *For any coherent sheaf S over M, $H^q(M, S) = 0$, $q \geqq 1$.*

The proof of the direct part of the theorem, i.e., that a Stein manifold has the properties (A) and (B), is difficult. The theorem has many consequences of which we mention the following:

(1) The additive Cousin problem always has a solution on a Stein manifold.

(2) The second Cousin problem, the problem whether a given divisor is the divisor of a meromorphic function, has a solution on a Stein manifold if $H^2(M, Z)=0$.

(3) Every meromorphic function on a Stein manifold is the quotient of two holomorphic functions.

Compact complex manifolds. When M is compact, Cartan and Serre proved that $H^q(M, S)$ is of finite dimension for any analytic coherent sheaf S.

Let M be of complex dimension m. Let W be a holomorphic vector bundle over M and let W^* be its dual bundle. Denote by $\Lambda^r(W)$ the sheaf of germs of holomorphic differential forms of degree r with values in W. Serre's duality theorem [**122**] says that the vector spaces

$$H^q(M, \Lambda^r(W)) \quad \text{and} \quad H^{m-q}(M, \Lambda^{m-r}(W^*))$$

are in duality and hence have the same dimension. Actually Serre's duality theorem is valid for noncompact complex manifolds; we state it for compact manifolds for simplicity.

By introducing an hermitian scalar product in the vector bundle W, Kodaira extended Hodge's harmonic forms and proved that $H^q(M, \Lambda^p(W))$ is isomorphic to the space $H^{p,q}$ of harmonic forms of type (p, q) and with values in W [**79**]. He, and later Akizuki and Nakano, gave sufficient conditions of a differential-geometric nature for the vanishing of the cohomology groups $H^q(M, \Lambda^p(W))$ (for the particular case of a line bundle W) [**2**], [**80**]. The method is a generalization of Bochner's method on riemannian manifolds.

9. **Characteristic classes** [**12**], [**25**], [**64**], [**96**]. A systematic theory of characteristic classes begins with the universal bundle theorem. It says that with a given compact manifold M as base space the fiber bundles with a structural group G (Lie group) are in one-one correspondence with the homotopy classes of mappings of M into a classifying space B_G; the latter depends only on G and on the dimension of M. Over B_G there is a universal bundle with the group G and the correspondence is established by the condition that the given bundle over M is induced by a mapping $f: M \to B_G$, which is defined up to a homotopy. With a coefficient ring A the induced homomorphism

$$f^*: H^*(B_G, A) \to H^*(M, A)$$

is completely determined by the bundle and is called the characteristic homomorphism. The image elements of this homomorphism are called the characteristic classes. The product bundle is induced by the constant mapping, for which all the characteristic classes except

1 are zero. Thus the characteristic classes are the first invariants which describe the deviation of a fiber bundle from a product bundle. From the definition it follows that the characteristic classes have the naturality property, i.e., that they are contravariant functors under bundle maps.

The first problem is to find the classifying spaces B_G for given groups G and to determine their cohomology rings. For $G=U(q)$, $B_{U(q)}$ can be chosen to be the Grassmann manifold $G(q, N, C)$ of all linear subspaces of dimension q through the origin of the complex Euclidean space $E_{q+N}(C)$ of dimension $q+N$, with N sufficiently large. Similarly, $B_{O(q)}$ can be chosen to be the Grassmann manifold $G(q, N, R)$ (resp. $\tilde{G}(q, N, R)$) of all linear subspaces (resp. oriented linear subspaces) of dimension q through the origin of the real Euclidean space $E^{q+N}(R)$ of dimension $q+N$, with N large. The cohomology groups of these Grassmann manifolds have been determined by Ehresmann. In particular, there are elements of $H^{2i}(G(q, N, C), Z)$, $0 \leqq i \leqq q$, which can, for instance, be completely described by the Schubert varieties and which generate the cohomology ring $H^*(G(q, N, C), Z)$, such that their images under f^* are the characteristic classes $c_i \in H^{2i}(M, Z)$ of the $U(q)$-bundle. Similarly, we have, for an $O(q)$-bundle or an SO(q)-bundle, the characteristic classes $W^i \in H^i(M, Z_2)$, $0 \leqq i \leqq q$, and $p_k \in H^{4k}(M, Z)$, $0 \leqq k \leqq [q/4]$, called respectively the Stiefel-Whitney classes and the Pontrjagin classes. A finer analysis of the cohomology of $G(q, N, R)$ allows a definition of W^{2i+1} as an element of $H^{2i+1}(M, Z)$. Also, for an SO(q)-bundle, the highest dimensional Stiefel-Whitney class W^q can be defined to be an element of $H^q(M, Z)$, and we will call it the Euler class.

From this definition one can immediately derive necessary conditions on characteristic classes for the reduction of the structural group of a fiber bundle. Let G' be a subgroup of G. The universal bundle $E_{G'} \to B_{G'}$ can be considered to be a G-bundle and is hence induced by a mapping $h: B_{G'} \to B_G$ of the respective classifying spaces. A bundle over M induced by a mapping $f: M \to B_G$ is a G'-bundle, if and only if a mapping f' exists such that the following diagram is commutative:

$$\begin{array}{ccc} M & \xrightarrow{f} & B_G \\ & f' \searrow \quad \nearrow h & \\ & B_{G'} & \end{array}$$

i.e., $f = h \circ f'$. In terms of the cohomology rings of these spaces this implies the commutativity of the diagram:

$$\begin{array}{ccc} H^*(M, A) & \xleftarrow{f^*} & H^*(B_G, A) \\ & f'^*\nwarrow \quad \swarrow h^* & \\ & H(B_{G'}, A) & \end{array} .$$

The latter generally implies relations between the characteristic classes of the G-bundle.

EXAMPLE 1. $G=\mathrm{SO}(q)$, $G'=\mathrm{SO}(q-1)$. The reduction of a G-bundle to a G'-bundle is in this case equivalent to the existence of a cross-section of the associated bundle of spheres $S^{q-1}=\mathrm{SO}(q)/\mathrm{SO}(q-1)$. To define the mapping

$$h: \tilde{G}(q-1, N, R) \to \tilde{G}(q, N, R)$$

geometrically we consider $\tilde{G}(q, N, R)$ to be the Grassmann manifold of oriented q-dimensional linear spaces through the origin of $E^{q+N}(R)$ as defined above and $\tilde{G}(q-1, N, R)$ to be that of oriented $(q-1)$-dimensional linear spaces through the origin of $E^{q+N-1}(R) \subset E^{q+N}(R)$. Let x_0 be a nonzero vector through the origin of $E^{q+N}(R)$ and perpendicular to $E^{q+N-1}(R)$. To an element of $\tilde{G}(q-1, N, R)$ its image under h is the linear space spanned by it and x_0. By studying this mapping h one derives easily that a necessary condition for the reduction of the G-bundle to a G'-bundle is that the Euler class should be zero. (In this case it can be proved that the condition is also sufficient.)

EXAMPLE 2. $G=\mathrm{SO}(2m)$, $G'=U(m)$. For the case of the tangent bundle of a manifold the reduction to a G'-bundle is equivalent to the existence of an almost complex structure. The mapping

$$h: G(m, N, C) \to G(2m, 2N, R)$$

is defined by taking, to an m-dimensional complex vector space of $E_{m+N}(C)$, its underlying real vector space. A study of the homomorphism on the cohomology rings induced by h gives the conditions [139]

$$W^i = 0, \qquad (i \text{ odd})$$
$$(39) \qquad \sum_{0 \leq i \leq [m/2]} (-1)^i p_i = \sum_{0 \leq i \leq m} (-1)^i c_i \sum_{0 \leq j \leq m} c_j,$$

where p_i are the Pontrjagin classes of M and c_i are the characteristic classes of the reduced $U(m)$-bundle. If $M=S^{4k}$, all the Pontrjagin classes will be zero and the formula gives $c_{2k}=0$. But c_{2k} can be proved to be the Euler class of the tangent bundle, so that we get a contradiction. This proves that S^{4k} has no almost complex structure.

These examples show the importance of finding relations between the characteristic classes of a G-bundle and those of its reduced G'-bundle, $G' \subset G$.

Further necessary conditions are found by considering the cohomology operations (Bockstein's operations, Steenrod's cohomology operations, Pontrjagin-Thomas operations, etc.). By using the for mulas expressing $p_p^i c_j$ as a polynomial of $c_1, \cdots, c_m$ (p_p^i is the for-rod reduced power operation), Borel and Serre proved that S^{2m} is not almost complex for $m \geqq 4$ [**26**].

Operations on vector bundles. Operations on fibers which commute with the action of the structural group can be extended to operations on bundles. For bundles with the group $\mathrm{GL}(q, R)$ or $\mathrm{GL}(q, C)$ (real or complex vector bundles), two such operations are particularly important: (1) Whitney sum; (2) tensor product. (There are other operations.) To define them consider $\mathrm{GL}(q)$ (over the real or complex field) to be the group of all $q \times q$ nonsingular matrices. Define

$$w: \mathrm{GL}(q_1) \times \mathrm{GL}(q_2) \to \mathrm{GL}(q_1 + q_2)$$

by

$$w(A_1, A_2) = \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix}, \quad A_i \in \mathrm{GL}(q_i), \quad i = 1, 2.$$

If over M there are $\mathrm{GL}(q_1)$ and $\mathrm{GL}(q_2)$ bundles which, relative to a covering $\{U, V, \cdots\}$, are given by the transition functions γ'_{UV}, γ''_{UV}, their Whitney sum has the transition functions $w(\gamma'_{UV}, \gamma''_{UV})$. Similarly, the tensor product of matrices defines a mapping

$$\mathrm{GL}(q_1) \times \mathrm{GL}(q_2) \to \mathrm{GL}(q_1 q_2),$$

from which one gets the tensor product of vector bundles.

It is possible to express the characteristic classes of a Whitney sum (resp. tensor product) in terms of those of the summands (resp. factors). The relations are particularly simple for complex vector bundles. For a $\mathrm{GL}(q, C)$-bundle W let

$$c(W) = 1 + c_1(W) + \cdots + c_q(W). \tag{40}$$

Then

$$c(W_1 \oplus W_2) = c(W_1)c(W_2), \tag{41}$$

where $W_1 \oplus W_2$ is the Whitney sum. This relation (41) is usually called the Whitney duality theorem. Write

$$c(W) = \prod_{1 \leq i \leq q} (1 + \gamma_i(W)). \tag{42}$$

Then the characteristic classes of a tensor product are given by the formula

$$c(W_1 \otimes W_2) = \prod_{1 \leq i \leq q_1;\, 1 \leq j \leq q_2} (1 + \gamma_i(W_1) + \gamma_j(W_2)). \tag{43}$$

Characteristic classes and curvature. The characteristic classes with real coefficients are closely related to the curvature of a connection. Such relations include some of the classical results on global differential geometry. Let $E \to M$ be a principal fiber bundle with a structural group which is compact and connected. Consider a real valued multilinear function $F(X_1, \cdots, X_s)$, with arguments X_i in the Lie algebra $L(G)$, such that it satisfies the conditions: (1) It is symmetric in any two arguments; (2) It is invariant under the action of the adjoint group, i.e.,

$$F(\mathrm{ad}(\gamma)X_1, \cdots, \mathrm{ad}(\gamma)X_s) = F(X_1, \cdots, X_s), \qquad \gamma \in G. \tag{44}$$

Suppose there be a connection in the bundle, with the curvature form Φ, which is $L(G)$-valued. To the arguments of F we substitute Φ, putting

$$F(\Phi) = F(\Phi, \cdots, \Phi). \tag{45}$$

Then $F(\Phi)$ is a closed exterior differential form of degree $2s$ in M. A theorem of Weil states that the cohomology class determined by $F(\Phi)$ is independent of the choice of the connection and depends only on the bundle [**40**]. In this way one can identify such functions F (which depends only on the group G) with the characteristic classes of the bundle.

EXAMPLE 3. $G = U(q)$. $L(U(q))$ is the space of all complex-valued $q \times q$ skew-hermitian matrices. Let $(\Phi_{ij}) = -{}^t(\overline{\Phi}_{ij})$ be the curvature form, and let

$$\det\left(\delta_{ij} + \frac{1}{2\pi\sqrt{-1}}\Phi_{ij}\right) = 1 + \Phi_1 + \cdots + \Phi_q, \tag{46}$$

where Φ_k is an exterior differential form of degree $2k$ in M. The cohomology class determined by Φ_k is the characteristic class c_k.

EXAMPLE 4. $G = \mathrm{SO}(2q)$. The integrand in the Gauss-Bonnet formula (2) is the Euler class.

Divisibility. For a $\mathrm{GL}(q, C)$-bundle the characteristic class c_k is an element of $H^{2k}(M, Z)$. If M is oriented and of even dimension $n=2m$, a polynomial of the form

(47) $$\sum_{k_1+\cdots+k_s=m} \alpha_{k_1\cdots k_s} c_{k_1} \cdots c_{k_s}$$

with the integer coefficients $\alpha_{k_1\cdots k_s}$ is an element of $H^{2m}(M, Z)$ and has an integral value over the fundamental homology class of M. Such an integer is called a characteristic number. An important divisibility property is given by the following theorem of Bott [**28**]: *The characteristic number c_n of any complex vector bundle over the sphere S^{2n} is divisible by $(n-1)!$.* This theorem has many geometrical applications. For instance, if the vector bundle has the tangent bundle of S^{2n} as the underlying real vector bundle, we would have $c_n=2$ = Euler-Poincaré characteristic of S^{2n}. This is imossible for $n \geqq 4$, which gives a new proof of the Borel-Serre theorem that S^{2n} has no almost complex structure for $n \geqq 4$.

Generally speaking, the characteristic classes c_k, being cohomology classes with integer coefficients, seem to characterize a complex vector bundle quite strongly. A theorem of Peterson says that if M has in dimension $2k$ only torsion coefficients which are relatively prime to $(k-1)!$, then a complex vector bundle of sufficiently large dimension over M is a product bundle if all its characteristic classes are zero [**110**].

10. **Riemann-Roch, Hirzebruch, Grothendieck, and Atiyah-Singer Theorems** [**11**], [**13**], [**27**], [**64**], [**108**]. The classical Riemann-Roch theorem for compact Riemann surfaces has received important developments through the recent works of Hirzebruch, Grothendieck, and Atiyah-Singer. Because of their relations with different fields of mathematics we will follow the historical development.

Let M be a compact Riemann surface and f a meromorphic function over M. At a point p with the local coördinate $z(z(p)=0)$ we have

(48) $$f = \alpha_n z^n + \alpha_{n+1} z^{n+1} + \cdots, \quad \alpha_n \neq 0.$$

The integer $n(p)$ is independent of the choice of the local coordinate and is called the order of f at p. The point p is a pole if $n(p)<0$. We call

(49) $$\operatorname{div}(f) = \sum_p n(p)p$$

the divisor of the meromorphic function f. The sum is a finite sum, because the zeros and poles are finite in number.

Generally a finite sum

$$D = \sum_p m(p)p, \qquad m(p) \in Z, \tag{50}$$

is called a divisor. The integer $d(D) = \sum m(p)$ is called its degree. All the divisors form an additive group. The divisor D is said to be positive ($\geqq 0$), if all $m(p) \geqq 0$.

The Riemann-Roch problem is to study the dimension of the complex vector space of all meromorphic functions f such that $\operatorname{div}(f) + D \geqq 0$, for a given divisor D. The latter condition means that f has a pole of order $\leqq +m(p)$ if $m(p) > 0$, and a zero of order $\geqq -m(p)$ if $m(p) < 0$, and is regular at all other points. We denote the dimension of the complex vector space in question by $l(D)$, and the classical Riemann-Roch theorem is given by the formula

$$l(D) = d(D) - g + i + 1, \tag{51}$$

where g is the genus of M and $i \geqq 0$ is the index of specialty of D.

The index of specialty i is defined as follows: First we say that two divisors D_1, D_2 are linearly equivalent, if there exists a meromorphic function f such that $D_1 - D_2 = \operatorname{div}(f)$. Since $\operatorname{div}(fg) = \operatorname{div}(f) + \operatorname{div}(g)$, $\operatorname{div}(f^{-1}) = -\operatorname{div}(f)$, linear equivalence is an equivalence relation. Let $\omega = b(z)dz$ be a meromorphic differential form. Locally we can write

$$b(z) = b_n z^n + b_{n+1} z^{n+1} + \cdots, \; b_n \neq 0.$$

Then $n(p)$ is called the order of ω at p, and we define the divisor of ω to be $\operatorname{div}(\omega) = \sum n(p)p$. The linear equivalence class of $\operatorname{div}(\omega)$, to be denoted by K, is independent of ω and depends only on M, because the ratio of two meromorphic differential forms is a meromorphic function. K is called the canonical divisor class. The index of specialty i is by definition

$$i = \dim H^0(M, \Omega(K - D)). \tag{52}$$

(For a divisor D, $\Omega(D)$ is the sheaf of germs of mermorphic functions f such that $\operatorname{div}(f) + D \geqq 0$.) By Serre's duality theorem we have the isomorphism

$$H^0(M, \Omega(K - D)) \cong H^1(M, \Omega(D)). \tag{53}$$

Then the Riemann-Roch theorem can be written

$$\dim H^0(M, \Omega(D)) - \dim H^1(M, \Omega(D)) = d - g + 1. \tag{54}$$

This formula was generalized by Hirzebruch to higher dimensions in the following way: Let M be a compact complex manifold of complex dimension m, and W an analytic vector bundle with the structural group $\mathrm{GL}(q, C)$. W is called a line bundle if $q=1$. (A line bundle is essentially the differential-geometric version of a divisor. For a divisor is given in a coordinate neighborhood U by an equation $\phi_U=0$, and it defines a line bundle with the transition functions $g_{UV}=\phi_U/\phi_V$. Conversely, it was proved by Kodaira and Spencer that every line bundle over a nonsingular algebraic variety is defined by a divisor.) Let $\Omega(W)$ be the sheaf of germs of holomorphic sections of W, and let

$$\chi(M, W) = \sum_{0 \leqq i \leqq m} (-1)^i \dim H^i(M, \Omega(W)). \tag{55}$$

Let c_i, $1 \leqq i \leqq m$, be the characteristic classes of the tangent bundle of M and d_j, $1 \leqq j \leqq q$, those of the bundle W. Write

$$\begin{aligned} 1 + \sum c_i &= \prod (1 + \gamma_i), \\ 1 + \sum d_j &= \prod (1 + \delta_j), \end{aligned} \tag{56}$$

and put

$$T(M, W) = \left\langle (e^{\delta_1} + \cdots + e^{\delta_q}) \prod_i \frac{\gamma_i}{1 - \exp(-\gamma_i)}, M \right\rangle, \tag{57}$$

where the right-hand side denotes the value of the cohomology class in question over the fundamental class of M, the value being zero for every summand of dimension $\neq 2m$. Hirzebruch's formula says that, for nonsingular algebraic varieties, we have

$$\chi(M, W) = T(M, W). \tag{58}$$

If W is not involved, then

$$\chi(M) = T(M). \tag{59}$$

$\chi(M)$ is called the arithmetic genus and $T(M)$ the Todd genus. If $m=1$ and W is a line bundle defined by a divisor D, then $d_1 \cdot M = d(D)$, $c_1 \cdot M = 2-2g$, and we get immediately the classical Riemann-Roch theorem from Hirzebruch's formula. Hirzebruch's proof of his formula makes use of Thom's theory of "cobordism."

Grothendeick formulates the Riemann-Roch problem as a problem on mappings. Let $F(M)$ be the free abelian group generated by the set of all isomorphy classes of complex vector bundles over M. M need not be connected, in which case it is not assumed that the bundle has the same fiber dimension over different components of M. An element $x \in F(M)$ is thus a formal finite linear combination

$$x = \sum n_i W_i, \qquad n_i \in Z, \quad W_i = \text{complex vector bundle}. \tag{60}$$

Let

$$0 \to W' \to W \to W'' \to 0$$

be a short exact sequence of complex vector bundles. Let $K(M)$ be the quotient group of $F(M)$ modulo the subgroup generated by $W-W'-W''$ for all short exact sequences. (In other words, we consider all extensions of complex vector bundles to be trivial. This is a natural idea, because by the Whitney duality theorem all the characteristic classes of $W-W'-W''$ are zero.) With the Whitney sum as addition and the tensor product as multiplication, $K(M)$ becomes a ring, called the Grothendieck ring.

Without danger of confusion, if W is a complex vector bundle, we will denote also by W the element it determines in $K(M)$. We define the character

$$\text{ch}(W) = \sum_{1 \leq j \leq q} e^{\delta_j} \in H^*(M, Q), \tag{61}$$

where $H^*(M, Q)$ is the cohomology ring of M with coefficients in the rational number field Q. Then

$$\text{ch}: K(M) \to H^*(M, Q) \tag{62}$$

is a ring homomorphism.

Generalizing the Todd genus we define the Todd cohomology class

$$t(M) = \prod_i \frac{\gamma_i}{1 - \exp(-\gamma_i)} \in H^*(M, Q). \tag{63}$$

Let f: $M \to N$ be a continuous mapping, where M, N are compact oriented manifolds of real dimensions μ, ν respectively. The diagram

$$\begin{array}{ccc} H_p(M, Q) & \xrightarrow{f_*} & H_p(N, Q) \\ \uparrow & & \uparrow \\ H^{\mu-p}(M, Q) & \xrightarrow{f_*} & H^{\nu-p}(N, Q) \end{array}$$

where the first row is the induced linear mapping on the homology vector spaces and the vertical rows are isomorphisms established by Poincaré's duality, defines a linear mapping of the last row, which we also denote by f_*.

Now let M, N be nonsingular algebraic varieties, and f: $M \to N$ be a holomorphic mapping. An additive homomorphism $f_!: K(M) \to K(N)$ can be defined as follows: Let $W \in K(M)$. To every open set $V \subset N$ assign the group $H^q(f^{-1}(V), \Omega(W))$. This is the presheaf of a sheaf

$R^q\Omega$. Define $f_!(W)=\sum_i(-1)^iR^i\Omega$. For algebraic variety, the algebraic coherent sheaves and complex vector bundles can be identified. Hence $f_!(W)\in K(N)$.

Grothendieck's theorem is then the formula:

$$f_*(\mathrm{ch}(W)t(M)) = \mathrm{ch}(f_!W)t(N). \tag{64}$$

Frow this Hirzebruch's formula follows by taking N to be a point. Another important consequence of the theorem is that the Todd genus of a nonsingular algebraic variety is a birational invariant.

A remark could be made as to the reason of the form of the factors in the Todd class. This form automatically appears when one considers the simple case of a hypersurface in projective space. From this one extends it to algebraic varieties which are complete intersections of hypersurfaces and then to the most general algebraic varieties.

Atiyah and Hirzebruch generalized a weakened version of Grothendieck's theorem to real manifolds. It starts with the observation that the Todd class can be written

$$t(M) = \exp(c_1/2)\alpha(M), \tag{65}$$

where

$$\alpha(M) = \prod \frac{\dfrac{\gamma_i}{2}}{\sinh\dfrac{\gamma_i}{2}} \in H^*(M, Q), \tag{66}$$

to be called the A-class, depends on the Pontrjagin classes of M only. Since $c_1\equiv W^2$, mod 2, for the Todd class to be defined the real manifold must have the property that its Stiefel-Whitney class W^2 is the reduction mod 2 of an integral class. A necessary and sufficient condition for this is that $W^3=0$. (Note that W^3 is a class with integer coefficients.)

Now let M be a real manifold with $W^3=0$. Let $d\in H^2(M, Z)$ be such that its reduction mod 2 is W^2. We call

$$R(M) = \mathrm{ch}(K(M))\exp(d/2)\alpha(M) \in H^*(M, Q) \tag{67}$$

the Riemann-Roch group of M. It is independent of the choice of d and is isomorphic to $\mathrm{ch}(K(M))$ as groups. A weakened version of Grothendieck's theorem states that

$$f_*(R(M)) \subset R(N), \tag{68}$$

if M, N are nonsingular algebraic varieties and $f: M \to N$ is a holomorphic mapping. The generalization by Atiyah-Hirzebruch is the following theorem.

Let M and N be compact oriented differentiable manifolds with $W^3(M) = W^3(N) = 0$ and $\dim M - \dim N \equiv 0$, mod 2. *Let $f: M \to N$ be a continuous mapping. Then $f_*(R(M)) \subset R(N)$.*

The theorem has various applications. We mention the following:

(1) The Todd genus of a compact almost complex manifold is an integer (Milnor).

(2) If $W^2(M) = 0$, then the A-genus is an integer.

(3) From their theorem Atiyah and Hirzebruch derived sharp lower bounds of the dimension of the sphere into which the real and complex projective spaces can be differentiably imbedded.

Atiyah and Singer considered elliptic differential operators on a compact (orientable) real differentiable manifold and obtained an important theorem which contains the Riemann-Roch-Hirzebruch formula (58) as a special case. To an elliptic differential operator D (from the sections of one complex vector bundle to those of another) one associates, through the character defined in (61), a cohomological invariant $\mathrm{ch}(D) \in H^*(M, Q)$, the character of D. Moreover, the definition of the Todd class in (63) can be extended to a real manifold by simply taking the complexification of the tangent bundle of M. It will then be expressible as a polynomial of the Pontrjagin classes of M. The rational number

$$i_t(D) = \langle \mathrm{ch}(D) t(M), M \rangle, \tag{69}$$

where the right-hand side denotes the pairing of a cohomology class and the fundamental class M, is called the topological index of the elliptic operator D.

On the other hand, D being an elliptic operator on a compact manifold, the spaces $\mathrm{Ker}(D)$ (i.e., the null space) and Coker (D) are both finite dimensional, and we define the analytical index of D to be the integer

$$i_a(D) = \dim \ker D - \dim \operatorname{coker} D. \tag{70}$$

The Atiyah-Singer theorem says that $i_t(D) = i_a(D)$.

Among the consequences of the Atiyah-Singer theorem are:

(1) The Gauss-Bonnet formula (2), when applied to the operator $d + \delta$, where δ is the codifferential defined by $\delta = \pm * d *$.

(2) The Hirzebruch index theorem, expressing the index $\tau(M)$ in terms of the Pontrjagin classes of M (cf. §5).

(3) Extension of the Riemann-Roch-Hirzebruch formula (58) to an arbitrary compact complex manifold.

It will be of great importance to find a theorem on proper maps of compact complex manifolds, which will imply as consequences both the Grothendieck and the Atiyah-Singer theorems.

11. **Holomorphic mappings of complex analytic manifolds [1], [45], [137].** A holomorphic mapping $f: V \rightarrow M$ of a complex manifold V into a complex manifold M is a continuous mapping which is locally defined by expressing the local coordinates of the image point as holomorphic functions of those of the original point.

I. *Compact manifolds.* (All manifolds in this part of the discussion are compact.)

The condition of a holomorphic mapping is so strong that it is generally not clear whether, for given manifolds V, M, such a mapping exists, which is not a constant. In fact, it is well known that if M is a complex torus and V satisfies the condition $H^1(V, R) = 0$, then every holomorphic mapping of V into M is constant. Similarly, let $h_{r0}(V)$ (resp. $h_{r0}(M)$) be the dimension of the vector space of holomorphic exterior differential forms of type $(r, 0)$ of V (resp. M). If $\dim V = \dim M$ and if the Jacobian of f is not identically zero, then $h_{r0}(V) \geqq h_{r0}(M)$. It follows that a nonconstant holomorphic mapping f exists from a Riemann surface (=one-dimensional complex manifold) V into a Riemann surface M only when $h_{10}(V) \geqq h_{10}(M)$, where the two quantities are now the genera of V and M respectively.

The celebrated theorems of Chow and Kodaira [**58**], [**81**] can be interpreted as assertions on holomorphic mappings. In fact, Chow's theorem implies that if V is compact, the image set $f(V)$ of a holomorphic mapping $f: V \rightarrow P_m$ (=projective space of dimension m) is an algebraic variety. Kodaira's theorem says that if V is a Kähler manifold whose fundamental two-form has rational periods, then there exists a holomorphic mapping $f: V \rightarrow P_m$ which is one-to-one.

Suppose that a holomorphic mapping $f: V \rightarrow M$ exists. An important problem is to find relations between the invariants (such as the characteristic numbers) of V, M and quantities which depend on the mapping (for instance, the degree when $\dim V = \dim M$). A classical relation of this nature is the Riemann-Hurwitz formula: Let V, M be Riemann surfaces with Euler characteristics $\chi(V)$, $\chi(M)$ respectively. Let $f: V \rightarrow M$ be a holomorphic mapping of degree d. Then we have

$$\chi(V) + w = d\chi(M), \tag{69}$$

where w is the index of ramification, i.e., the sum of the orders of the points of ramification.

Another set of relations of this nature consists of the Plücker formulas for an algebraic curve. Let an algebraic curve be defined by a holomorphic mapping $f: V \to P_m$, where V is a Riemann surface. Suppose that the algebraic curve is nondegenerate, i.e., that the image $f(V)$ does not belong to a linear space of dimension $\leqq m-1$. To this curve is defined the pth associated curve $f^p: V \to \mathrm{Gr}(m, p)$, $0 \leqq p \leqq m-1$, formed by the osculating projective spaces of dimension p, where $\mathrm{Gr}(m, p)$ is the Grassmann manifold of all p-dimensional projective spaces in P_m (thus $\mathrm{Gr}(m, 0) = P_m$; notice the relation with the Grassman manifolds in §9 and the difference of notations). $f^p(V)$ defines a cycle in $\mathrm{Gr}(m, p)$, which is homologous to a positive integral multiple ν_p of the fundamental two-cycle of $\mathrm{Gr}(m, p)$. The integer $\nu_p > 0$ is called the order of rank p of our algebraic curve. Geometrically it is the number of points of the curve at which the osculating spaces of dimension p meet a generic linear space of dimension $m-p-1$ of P_m. A stationary point of order p is one at which the pth associated curve has a tangent with a contact of higher order. The stationary points are isolated and a positive index can be associated to each of them. Let w_p be the sum of indices at the stationary points of order p. Then Plücker's formulas are

$$(70) \qquad -w_p - \nu_{p-1} + 2\nu_p - \nu_{p+1} = \chi(V), \qquad 0 \leqq p \leqq m-1.$$

Here the right-hand side is an invariant of V, while the left-hand side involves quantities which depend on the mapping. The special cases of the Plücker formula for $m=1$ and the Riemann-Hurwitz formula for $M = P_1$ give the same relation, as to be expected.

For nonsingular algebraic varieties a much more profound relation between invariants of manifolds and quantities depending on a holomorphic mapping is given by Grothendieck's Riemann-Roch theorem (cf. §10). Here one utilizes the classification of singularities by Thom. Applying the results of Grothendieck and Thom, I. R. Porteous **[111]** derived relations between the characteristic classes for the following cases: (a) dilatations; (b) ramified coverings with singularities of a relatively simple type.

If V and M are of the same dimension, an important invariant of the mapping f is its degree $d(f) \geqq 0$. It is equal to the number $n(\alpha)$ of times that any point $\alpha \in M$ is covered by the image of the mapping. Define an hermitian metric on M such that the total volume of M is 1. Then we have

$$(71) \qquad d(f) = n(\alpha) = \operatorname{vol} f(V).$$

II. *Equidistribution.* The most important case of a holomorphic mapping $f: V\rightarrow M$ is when V is the euclidean line E_1 and M is the projective line P_1, in which case the mapping is nothing else but a meromorphic function. The famous theorem of Picard says that if f is not a constant, the set $P_1-f(E_1)$ cannot contain more than two points. Much sharper results generalizing Picard's theorem were obtained by R. Nevanlinna in the form of his defect relations. The general aim is to ascertain, whenever it is true, that the image set is large and that, in a suitable sense it is "equidistributed."

As with meromorphic functions, the case which will yield interesting results is when V is noncompact and M is compact. This assumption we will make throughout Part II except in the last section. The basic idea is the realization that results which are true for compact V can be extended to those V, which are "complex analytically large" (such as a parabolic open Riemann surface or a high-dimensional domain with a pseudo-concave exhaustion). The manifold V is to be exhausted by a sequence of compact polyhedra with boundaries, so that an intermediate step is to study the mapping $f: D\rightarrow M$ where D is a compact polyhedron with boundary. Generalizations of the results of Part I to this case are the necessary prerequisites. What remains consists in the estimate and the study of the asymptotic behavior of the boundary integrals, which are usually delicate considerations.

So far the deepest result in this direction is a theorem of Ahlfors on holomorphic curves in projective space, whose study was initiated by H. and J. Weyl. By a holomorphic curve is meant a holomorphic mapping $f: V\rightarrow P_m$ (our holomorphic curve is the same as a meromorphic curve in Weyl's terminology). It is said to be nondegenerate, if $f(V)$ does not belong to a linear subspace of P_m. Ahlfors' theorem, in a slightly generalized form, is the following:

Let $f: V\rightarrow P_m$ be a nondegenerate holomorphic curve such that V can be compactified as a Riemann surface by the addition of a finite number of points. Given

$$\binom{m+1}{p+1}+1$$

linear spaces of dimension $m-p-1$ in general position, one of them must meet an osculating space of dimension p of the curve.

Considering the beauty of the theorem, it is natural to ask whether the image space P_m can be replaced by a more general space, such as the Grassmann manifold $\mathrm{Gr}(m, p)$ of all p-dimensional (projective) linear subspaces in P_m $(0<p<m)$. By imbedding $\mathrm{Gr}(m, p)$ in P_N, with

$$N = \binom{m+1}{p+1} - 1,$$

one derives from the Ahlfors' theorem results of the Picard type on the relative position of a holomorphic curve in $\mathrm{Gr}(m, p)$ with respect to a finite number of prime divisors in general position on $\mathrm{Gr}(m, p)$. However the bound so obtained is generally not sharp. To put ourselves in a specific position we would like to ask the following question: Suppose $f: C \to \mathrm{Gr}(2m-1, m-1)$ be a holomorphic curve. To each $\zeta \in C$ let $f(\zeta)$ be spanned by m points whose homogeneous coordinate vectors are $Z_1, \cdots, Z_m$. The curve is called nondegenerate, if the determinant

$$\left(Z_1, \cdots, Z_m, \frac{dZ_1}{d\zeta}, \cdots, \frac{dZ_m}{d\zeta}\right)$$

does not vanish identically. Is it true that, given $2m+1$ linear subspaces of dimension $m-1$ of P_{2m-1} in general position, one of them will intersect $f(\zeta)$ for a certain $\zeta \in C$? The affirmative answer to this for $m=1$ is the classical Picard theorem.

When dim $V > 1$, value distribution in the strict sense fails completely. In fact, the classical example of Fatou and Bieberbach is a holomorphic mapping $f: E_2 \to E_2$, such that the Jacobian determinant is identically equal to one, while $E_2 - f(E_2)$ contains an open subset. The problem should thus be looked at from a more general viewpoint.

When dim $V = n$, and $M = P_m$, $n \leqq m$, the first main theorem can be generalized, as follows: Let A be a linear subspace of dimension $m-n$ in P_m. Let $A^\perp$ be the linear subspace of dimension $n-1$ which is orthogonal to A (relative to the standard hermitian metric in P_m). To $Z \in P_m - A$, the space spanned by Z and A meets $A^\perp$ in a unique point. This defines a mapping $\psi: P_m - A \to A^\perp$. Denote by $s(Z, A)$ the distance between Z and A. Let Ω be the fundamental two-form of P_m, so normalized that $\int_{P_m} \Omega^m = 1$. Let $j: A^\perp \to P_m$ be the identity mapping, and let $\Phi = (j \circ \psi)^* \Omega$. Then

$$(72) \qquad \Lambda = \frac{1}{2\pi i}(d' - d'') \log \sin s(Z, A) \wedge \left(\sum_{0 \leq k \leq n-1} \Phi^k \wedge \Omega^{n-k-1} \right)$$

is a real differential form of degree $2n-1$ in $P_m - A$. If A is generic, $f(D)$ cuts A in a finite number $n(A, D)$ of points, and we have the formula

$$(73) \qquad n(A, D) - v(D) = \int_{f(\partial D)} \Lambda,$$

provided that $f^{-1}(A)\cap\partial D=\emptyset$ ($v(D)$ is the volume of $f(D)$ and is equal to $\int_{f(D)}\Omega^n$). This formula, to be called the first main theorem, is due to H. Levine [92].

It turns out that this first main theorem leads to a geometrical conclusion, in the special case that V is the coordinate space of n dimensions, with the coordinates $z_1, \cdots, z_n$. We exhaust V by the balls D_r defined by $z_1\bar{z}_1+\cdots+z_n\bar{z}_n\leqq r^2$. Relative to the parameter r the first main theorem can be integrated. Let

$$(74)\quad v_k(r) = \int_{D_r} f^*\Omega^{n-k} \wedge \Omega_0^k, \quad 0 \leqq k \leqq n, \qquad \Omega_0 = \frac{i}{2}\sum_{1\leq l\leq n} dz_l \wedge d\bar{z}_l.$$

By applying integral geometry to the integrated form of the first main theorem, we obtain the following theorem [46]:

Let f: $V\to P_m$ be a holomorphic mapping of the euclidean space V of dimension n into P_m. Assume that the following conditions are fulfilled, as $r\to\infty$:

(1) *The order function*

$$T(r) = \int_{r_0}^{r} \frac{v_0(t)\,dt}{t^{2n-1}} \to \infty,$$

$$(2)\qquad \int_{r_0}^{r} (v_1'(t)\,dt)/t^{2n} = o(T(r)).$$

Then the set of linear spaces of dimension $m-n$ which do not meet $f(V)$ is of measure zero in the Grassmann manifold Gr$(m, m-n)$ *of linear subspaces of dimension $m-n$ in P_m, $m\geqq n$.*

A special but important case of holomorphic mappings is that of the holomorphic sections of a holomorphic vector bundle. The study of the zeroes of a section or of points at which several sections are linearly dependent is closely related to the value distribution problem and is in many cases equivalent to it. The problem is to describe those properties which depend on the bundle only and are independent of the sections, for the case that the base manifold is not necessarily compact. The algebraic aspect was recently studied by Bott and Chern [30], namely the properties of the differential forms in the vector bundle relative to the operator $D=id'd''$. This could be regarded as a refinement of the theory of characteristic classes for the complex analytic case. From it an equidistribution theorem was derived, but much remains to be done.

12. **Isometric mappings of Riemannian manifolds.** A mapping of a riemannian manifold into another is called an isometry, if it preserves

the lengths of the tangent vectors. If an isometry is present, the basic local invariant is the second fundamental form, which is a quadratic differential form, with value in the normal bundle. The second fundamental form is related to the curvature forms of both riemannian manifolds by the (generalized) Gauss equations, but until now the exact relationship (a purely algebraic problem) has not been sufficiently clarified.

The problem of isometric mappings is almost as old as differential geometry itself, beginning with the theory of curves and surfaces. If V and M are real analytic riemannian manifolds of dimensions n and $m=n(n+1)/2$ respectively, then the theorem of Janet-Cartan says that there is locally an isometric mapping of V into M [**35**]. Without analyticity the same result has only been proved for $n=2$, and in fact only under the additional assumption that the gaussian curvature of V keeps a constant sign in a neighborhood [**61**]. For $n>2$ the generalization of the Janet-Cartan theorem to C^∞-data seems to be difficult, even allowing additional conditions in the form of inequalities between the curvatures.

Perhaps the first global isometric imbedding theorem is the solution of the Weyl problem, which is to find an isometric imbedding of a two-dimensional compact riemannian manifold with positive gaussian curvature into the three-dimensional euclidean space [**138**]. Weyl's work was completed by H. Lewy, L. Nirenberg, A. D. Alexandrow, and A. V. Pogorelov [**3**], [**49**], [**91**], [**104**]. A bold isometric imbedding theorem of an arbitrary riemannian manifold into an euclidean space was proved by J. Nash. For a C^1-riemannian manifold V of dimension n Nash's theorem, improved by N. H. Kuiper, says that V can be C^1-isometrically imbedded in an euclidean space of dimension $2n+1$ [**85**], [**99**]. Among other results Kuiper also found an isometric imbedding of the hyberbolic plane as a closed subset in euclidean three-space. However, such imbeddings could be pathological. Nash's imbedding theorem is still true, if more smoothness is imposed, but the dimension of the receiving euclidean space will have to be higher and the proof is more difficult [**89**], [**97**], [**100**]. For example, Nash proved that a compact two-dimensional riemannian manifold can be isometrically imbedded in an euclidean space of dimension 17. Whether this bound can be improved should be a very interesting question in differential geometry.

If a riemannian manifold can be isometrically imbedded in an euclidean space E^m of dimension m there arises the question as to how far it is determined, modulo the isometries in E^m. In the general case practically nothing is known. A classical theorem of Cohn-

Vossen, complemented by a remark of K. Voss, says that a closed convex surface (i.e., Gaussian curvature $\geqq 0$) in an euclidean three-space E^3 is completely determined by its metric, modulo the isometries of E^3 [**131**].

To restrict the submanifolds under consideration and thus to make a Cohn-Vossen type uniqueness theorem more plausible, the notion of total absolute curvature seems to deserve attention. This is the volume of the image of the unit normal bundle of the submanifold under the Gauss mapping, so normalized that the volume of the unit sphere in E^m is 1. Chern and Lashof proved that a compact immersed submanifold in E^m has total curvature $\geqq 2$, and if it is equal to 2, it is a convex hypersurface imbedded in an euclidean space of one dimension higher [**44**]. For a given compact manifold the minimum of the total curvatures for all possible immersions is a differential topological invariant and is an integer. The so-called tight immersions, i.e., those for which the minimum total curvature is attained, can thus be considered as generalizations of convex hypersurfaces. It is not known whether there exist two tightly immersed compact submanifolds, which are isometric but which do not differ by an isometry of E^m.

A. D. Alexandrow proved that two tightly imbedded analytic compact surfaces of genus one in euclidean three-space E^3 differ by an isometry of E^3, if they are isometric. To remove the assumption of analyticity Nirenberg has to put in some additional hypotheses [**105**], so that the question whether the analyticity assumption in Alexandrow's theorem could be removed remains unsettled. On the other hand, the notion of a tight imbedding can be easily extended to polyhedral submanifolds, and T. Banchoff gave examples of two polyhedral surfaces of genus one in euclidean three-space E^3 such that their corresponding faces are congruent, but they do not differ by a motion or a reflection of E^3 [**14**].

An important class of submanifolds in E^m consists of the minimal submanifolds. This is an isometric immersion $V \rightarrow E^m$ such that the coordinate functions are harmonic functions on V. Equivalently, we can also say that the trace vector of the second fundamental form is identically zero or that it solves locally the Plateau problem, being the submanifold of the least area with a given boundary. A complete minimal submanifold in E^m is never compact. It is, however, not known whether it is unbounded. So far the study has been mostly confined to minimal surfaces (i.e., dimension two); the reason lies in the intimate relationship of this case to complex function theory. In fact, the surface, with a fixed orientation, has an underlying complex structure, and the Grassmann manifold of all oriented planes through a fixed point 0 of E^m is a homogeneous complex manifold. A surface in E^m is minimal, if and only if the Gauss mapping, which sends a point p of the surface to the plane through 0 parallel to the tangent plane at p, is anti-holomorphic. An important problem is to determine the type of the surface V (in the sense of complex function theory) from the properties of the Gauss mapping. A classical theorem of S. Bernstein, as generalized by R. Osserman, can be interpreted as a uniqueness theorem. It says that a complete minimal surface in E^3 is a plane, if its image under the Gauss mapping omits an open subset of the Grassmann manifold (which in this case is the unit sphere). This theorem has a generalization to minimal surfaces in E^m [**47**], [**107**].

13. **General theory of G-structures** [**32**], [**34**], [**37**], [**77**], [**126**]. The general theory of G-structures is concerned primarily with a local problem, i e., the equivalence problem formulated in §4. The first local invariant is the first-order structure function, whose definition is a little complicated and we will not give it here. A G-structure is called locally flat, if it is equivalent to one given by the differentials of the local coordinates ($\theta_U^i = du^i$ in the notation of §4). The structure function is zero for a locally flat structure. In some cases the converse of this statement is true. For an almost complex structure the vanishing of the structure function is equivalent to the fulfillment of the integrability conditions. That this implies local flatness is an easy theorem in the real analytic case. Without analyticity this is a theorem of Newlander and Nirenberg, whose proof involves delicate considerations in partial differential equations [**101**], [**103**].

Given a G-structure, its local automorphisms may depend on a finite number of constants or on arbitrary functions. The G-structure is then said respectively to be of finite or infinite type. For an irreducible linear group G, Kobayashi and Nagano found a necessary and sufficient condition for a G-structure to be of finite type in terms of their theory of filtered Lie algebras [**77**]. The case of infinite type leads to structures defined by infinite pseudo-groups. In this respect Cartan's determination of the simple infinite pseudo-groups (over the complex field) has not been completely verified, and its clarification should be one of the outstanding problems in the theory of infinite pseudo-groups.

Within this framework is a problem studied by Weyl and Cartan on the so-called Pythagorasian nature of metric. This is to find those G such that a unique affine connection exists, which preserves the admissible frames and which has a given torsion form. For dimensions $\geqq 3$ this implies that G is the orthogonal group (of arbitrary signature). The problem has recently been studied and extended by Klingenberg and Kobayashi-Nagano [70], [76].

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