ROLLING

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If k is a tame arc in the 3-dimensional half-space $R^3_+ = (x, y\ z, t: z \geqq 0, t = 0)$ that spans the plane $R^2 = (x, y, z, t: z = 0, t = 0)$ then a locally flat 2-sphere S in the 4-dimensional space $R^4 = (x, y, z, t:)$ is generated by k when R^3_+ is rotated about R^2. Nowadays the sphere S is said to be derived from k by *spinning*. By knotting k in various ways, various types of knotted spheres S can be obtained [1], but it is known that not every type of (locally flat) knotted sphere can be so obtained [2].

Some years ago I considered spheres S that are obtained from k by combining the spinning process with a simultaneous rotation of k about its "axis" (in R^3_+). This operation has come to be known as *twist-spinning*. The specific question that I raised at that time—whether the sphere obtained by twist-spinning a trefoil 3_1 (using a simple twist) is actually knotted—has been answered (in the negative) recently by C. Zeeman [3].

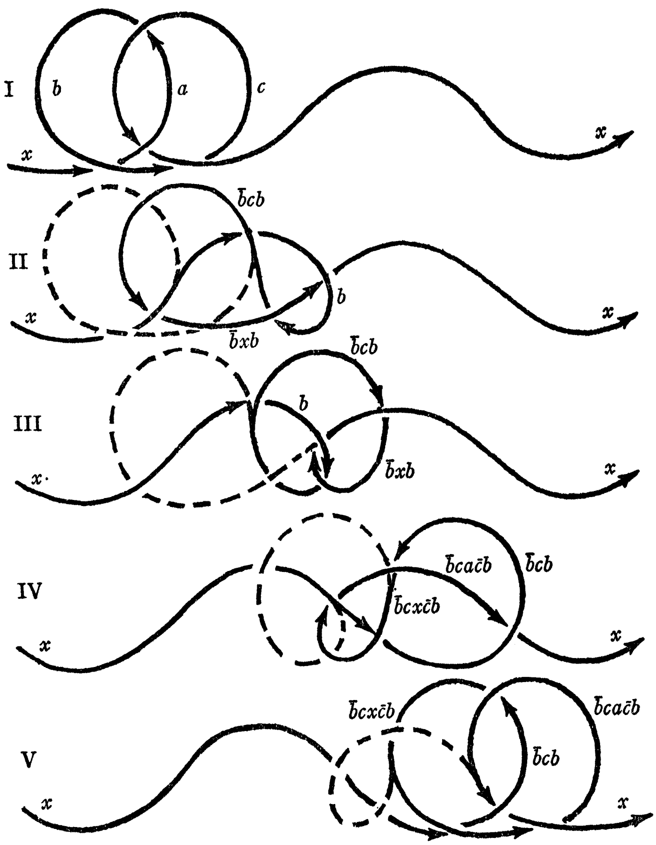
In this note I want to introduce another variation of the spinning process, one that I call *roll-spinning*. It is the same as twist-spinning except that instead of twisting, i.e. rotating k about its axis in R^3_+, I *roll* the knot along its axis. This operation (whose name derives from its resemblance to the operation of "rolling a stocking") is somewhat difficult to describe in totally precise terms, and I will content myself here with referring to Figure II, in [3], where it is shown how to roll a figure-eight knot 4_1.

My objective is to show that roll-spinning is not just twist-spinning in disguise (a state of affairs that one might suspect to be so). Specifically I shall show that a *simple roll-spin of* 4_1 *produces a type of knotted sphere* S *that cannot be obtained from* 4_1 *by any twist-spin.*

Figure I gives a projection of 4_1 with the meridian elements of its group G indicated by x, a, b, c. From this figure the presentation

$$(x, a, b, c: ab = bx, cb = ac, cx = xb)$$

is read off in the usual way [2], [4]. If we give 4_1 the twist-spin in which 4_1 is rotated about its axis n times the group Γ_n of the resulting sphere (cf. [2], [3]) has presentation



$$(x,\ a,\ b,\ c\colon ab = bx,\ cb = ac,\ cx = xb,\ x^n c = cx^n)$$

which simplifies to

$$(x,\ c\colon xcx^{-1}cx = cxc^{-1}xc,\ x^n c = cx^n).$$

The first elementary ideal (cf. [4]) is therefore

$$\gamma_n = (1 - 3t + t^2,\ 1 - t^n).$$

Figures I . . . V give a sequence of projections of 4_1 that show the successive stages of a simple "rolling." From this we obtain by the same method [2], [3] the following presentation of the group G_1 of the resulting knotted sphere:

$$(x,\ a,\ b,\ c\colon ab = bx,\ cb = ac,\ cx = xb,$$
$$b^{-1}cb = a,\ b^{-1}cxc^{-1}b = b,\ b^{-1}cac^{-1}b = c)$$

which simplifies to

$$(x,\ c\colon cxc = xcx,\ x^3 = c^3).$$

In this case the first elementary ideal is

$$g_1 = (1 - t + t^2,\ 2).$$

To show that $g_1 \neq \gamma_n$, we have only to map the ring of L-polynomials in t into the ring of integers of the cyclotomic field $K(\omega)$, where $\omega = e^{2\pi i/3}$, by $t \rightarrow \omega$, and note that

$$\begin{aligned} g_1 &\rightarrow (2), & \\ \gamma_n &\rightarrow (4) & \text{if } n \equiv 0 \pmod 3 \\ &\rightarrow (2,\ 1 - \omega) & \text{if } n \equiv 1 \pmod 3 \\ &\rightarrow (2,\ 1 - \omega^2) & \text{if } n \equiv 2 \pmod 3). \end{aligned}$$

To put the above considerations into proper perspective one should consider the space whose elements are the arcs k in R^3_+ that span R^2. Then twist- and roll-spinning appear as loops in this "configuration space." Thus we are led to consider *relative (2-dimensional) braid groups*. From the methodological point of view initiated in [5], this concept generalizes the ordinary (i.e. absolute 1-dimensional) braid groups. Absolute 2-dimensional braid groups were considered in [6], and the relative 2-dimensional braid groups that I have indicated here will be the subject of a more systematic study that C. H. Giffen is undertaking.

Bibliography

1. E. Artin, *Zur Isotopie zwei dimensionaler Flächen in R_4*, Abh. Math. Sem. Univ. Hamburg 4 (1925), 47–72.

2. R. H. Fox, *A quick trip through knot theory*, Proc. 1961 Top. Inst., Prentice-Hall, Englewood Cliffs, N. J., 1962; pp. 120–167.

3. E. C. Zeeman, *On twisting spun knots*, Trans. Amer. Math. Soc. **115** (1965), 471–495.

4. R. H. Crowell and R. H. Fox, *Introduction to knot theory*, Ginn, Boston, 1963.

5. R. H. Fox and L. Neuwirth, *The braid groups*, Math. Scand. **10** (1962), 119–126.

6. D. Dahm, *A generalization of braid theory*, Ph.D. Thesis, Princeton University, Princeton, N. J., 1962.

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