THE STRICT TOPOLOGY AND COMPACTNESS IN THE SPACE OF MEASURES

BY JOHN B. CONWAY[1]

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The strict topology β on the space $C(S)$ of bounded complex valued continuous functions on a locally compact space S was introduced by R. C. Buck [1] and has been studied by Glicksberg [4] and Wells [9]. Among the problems in mathematics which have seen successful applications of the strict topology are various ones in spectral synthesis (Herz [6]) and spaces of bounded analytic functions (Shields and Rubel [8]). In spite of these successes there has as yet been no detailed investigation of the relationship between this topological vector space and its adjoint. This is an announcement of some results from an attempt at such an investigation. The complete proofs, as well as those of some additional results and extensions of the present theorems, will appear elsewhere.

In particular we are interested in a question posed by Buck. Is $C(S)$ with the strict topology a Mackey space? As yet no characterization of those spaces S for which the answer is affirmative is available. However, we can prove a much stronger result whenever S is paracompact—a class which includes all σ-compact spaces and topological groups.

In the following $C_0(S)$ will denote the subspace of $C(S)$ consisting of all functions which vanish at infinity, and $C_c(S)$ those which vanish off some compact set. If $\phi \in C(S)$ then let $N(\phi)=\{s:\phi(s)\neq 0\}$, spt $(\phi)=N(\phi)^-$ (the closure of $N(\phi)$), $\|\phi\|_\infty=\sup\{|\phi(s)|:s\in S\}$, and $V_\phi=\{f\in C(S):\|\phi f\|_\infty\leqq 1\}$.

The *strict topology* β on $C(S)$ is defined by the neighborhood basis at the origin consisting of the sets $\{V_\phi:\phi\in C_0(S)\}$. Some of Buck's results are that $C(S)_\beta$ is complete, the β-bounded and norm bounded subsets of $C(S)$ are the same, $C_c(S)$ is β-dense in $C(S)$, and the adjoint of $C(S)_\beta$ is $M(S)$, the space of bounded regular Borel measures on S.

We will denote by "β-weak *" the weak star topology on $M(S)$

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which it has at the adjoint of $C(S)_\beta$, in order to distinguish it from the weak * topology which $M(S)$ has as the adjoint of the Banach space $C_0(S)$.

The remaining notation will be standard as is found in [3] and [7].

A set $H\subset M(S)$ is β-equicontinuous if and only if there exists a $\phi\in C_0(S)$ such that $H\subset V_\phi^0\equiv\{\mu\in M(S): |\int f d\mu|\leqq 1 \text{ for all } f\in V_\phi\}$. Therefore a characterization of the sets V_ϕ^0 is a necessity and is given by the following theorem of Glicksberg [4]. His proof is, however, quite complicated, and we will sketch a simple one.

THEOREM 1. *If $\phi\in C_0(S)$, then $V_\phi^0=\{\mu: \mu$ vanishes off $N(\phi)$ and $\|\mu/\phi\|\leqq 1\}$.*

PROOF. Define the map $T_\phi: C(S)_\beta\to C_0(S)$ by $T_\phi(f)=\phi f$. Let B^* = the unit ball of $M(S)$. Then T_ϕ is continuous and, if $T_\phi^*: M(S)\to M(S)$ is its adjoint, it is easy to see that $V_\phi^0=T_\phi^*(B^*)=\{\phi\nu: \|\nu\|\leqq 1\}$. It now quickly follows that V_ϕ^0 has the required form and the proof is complete.

It should be realized that functions of the form $1/\phi$ are not very manageable. Hence, an alternate characterization of β-equicontinuity which avoids this is desirable, and is given by the following result.

THEOREM 2. *A set $H\subset M(S)$ is β-equicontinuous if and only if (1) H is uniformly bounded and (2) for every $\epsilon>0$ there is a compact set $K\subset S$ such that $|\mu|(S\backslash K)\leqq\epsilon$ for all $\mu\in H$.*

The weak topology on $M(S)$ is the same whether we consider $M(S)$ as $C_0(S)^*$ or $C(S)_\beta^*$, because the strong topology on $C(S)_\beta^*$ is exactly the norm topology; i.e., $C_0(S)^{**}=C(S)_\beta^{**}$. In particular, if S is the space of positive integers then $C(S)=l^\infty$, $M(S)=l^1$, and $(l^\infty,\beta)^{**}=l^\infty$. Thus the β-weak * and weak topologies on l^1 are the same. By using the Eberlein-Smulian Theorem [3, p. 430] and the well-known fact that in l^1 weak and norm convergence of sequences is the same, we arrive at the following theorem.

THEOREM 3. *A subset $H\subset l^1$ is β-equicontinuous if and only if H is β-weak * (weakly) countably compact (i.e., every sequence in H clusters β-weak * to an element of l^1).*

We are now in a position to prove the principal theorem of this note.

THEOREM 4. *If S is paracompact and $H\subset M(S)$ is β-weak * countably compact then H is β-equicontinuous. Consequently $C(S)_\beta$ is a Mackey space.*

PROOF. We sketch the proof when S is σ-compact. Let $S=\bigcup_{n=1}^{\infty} D_n$ where D_n is compact and $D_n\subset \text{int } D_{n+1}$ (the interior of D_{n+1}). If H is not β-equicontinuous then we may find an $\epsilon>0$ and a sequence $\{(\mu_n, \phi_n, K_n, U_n)\}_{n=1}^{\infty}$ having the following properties: (a) $\mu_n\in H$, $\phi_n\in C_c(S)$, K_n is compact, U_n is open in S with U_n^- compact, $U_n^-\cap K_n=\square$, $\|\phi_n\|_\infty=1$, and spt $(\phi_n)\subset U_n$; (b) $D_n\cup K_n\cup U_n^-\subset \text{int } K_{n+1}$; (c) $|\mu_n(U_n)|>\epsilon/4$; (d) $|\mu_n|(U_n)<|\int\phi_n d\mu_n|+\epsilon/8$.

From these conditions it follows that $S=\bigcup_{n=1}^{\infty} \text{int} K_n$, $F=\bigcup_{n=1}^{\infty} \text{spt }(\phi_n)$ is closed in S, and if $x=\{x^{(n)}\}_{n=1}^{\infty}\in l^\infty$ then $f_x(s)=\sum_{n=1}^{\infty} x^{(n)}\phi_n(s)$ defines an element of $C(S)$ with $\|f_x\|_\infty=\|x\|_\infty$. Hence $T: l^\infty\to C(S)$ defined by $T(x)=f_x$ is a linear isometry. Furthermore, T: $(l^\infty, \beta)\to C(S)_\beta$ is continuous, and therefore if T^*: $M(S)\to l^1$ is its adjoint then $T^*(H)$ is β-weak * countably compact in l^1. Using Theorem 2 for l^1 and condition (d), we now obtain a contradiction to condition (c) on our sequence. This completes the proof.

This same method proves the general case if one chooses the sets K_n in a prudent manner. However this method cannot work if S is a pseudocompact noncompact space.

THEOREM 5. *If Ω_0 is the space of ordinals less than the first uncountable ordinal with the order topology then $C(\Omega_0)_\beta$ is not a Mackey space.*

PROOF. Let H be the β-weak * closed convex circled hull of all the measures of the form $\frac{1}{2}[\delta_{(s)}-\delta_{(s+1)}]$, where $\delta_{(s)}$ is the unit point mass at s, s is a nonlimit ordinal, and $s+1$ is its immediate successor. Then H is β-weak * compact but is not β-equicontinuous.

Let us close by pointing out some applications of our main theorem. In a paper by J. Dieudonné [2] several modes of sequential convergence in $M(S)$ are discussed when S is compact. More precisely, if $\{\mu_n\}$ and μ are in $M(S)$ and Φ is a given class of functions, all integrable with respect to $|\mu|$ and $|\mu_n|$ for $n\geqq 1$, he gives a condition C such that $\int\phi d\mu_n\to\int\phi d\mu$ for all $\phi\in\Phi$ if and only if $\{\mu_n\}$ satisfies condition C and $\mu_n\to\mu$ weak *. We can improve his results in three ways. First by using the concept of β-equicontinuity we can generalize all these theorems to locally compact spaces (note that β-equicontinuity implies that S is "approximately compact" with respect to the measures involved). Also, most of the arguments used in the necessity proofs of condition C have a noticeable similarity with one another, and in every case this necessity proof can be achieved by a judicious application of Theorem 4. Finally, we are able to improve some of Dieudonné's results. As an example we state the following theorem (see [2, pp. 32 and 35]; also see [5, p. 150]).

THEOREM 6. *If $\{\mu_n\}$, μ are in $M(S)$ then the following are equivalent:*

(a) $\mu_n \to \mu$ *weakly*; (b) $\{\mu_n\}$ *is uniformly bounded and* $\mu_n(U) \to \mu(U)$ *for every open set* U; (c) (i) $\mu_n \to \mu$ β*-weak* *, (ii) $\{\mu_n\}$ *is* β*-equicontinuous, and* (iii) *for every* $\epsilon > 0$ *and every compact set* $K \subset S$ *there is an open set* $V \supset K$ *such that* $|\mu_n|(V \backslash K) \leqq \epsilon$ *for all* $n \geqq 1$.

From this result we can prove a characterization of weak compactness due to Grothendieck [5, p. 146].

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Louisiana State University