CONSTRUCTION OF GLOBALLY CONVERGENT ITERATION FUNCTIONS FOR THE SOLUTION OF POLYNOMIAL EQUATIONS

BY J. F. TRAUB

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Iteration functions for the approximation of zeros of a polynomial P are usually given as explicit functions of P and its derivatives. We introduce a class of iteration functions which are themselves constructed according to a certain algorithm given below. The construction of the iteration functions requires only simple polynomial manipulation which may be performed on a computer.

Let P be a real monic polynomial of degree n with distinct zeros $\rho_1, \cdots, \rho_n$ and let the dominant zero ρ_1 be real. The theory may be extended to multiple zeros, dominant complex zeros, and subdominant zeros.

Let $B(t)$ be an arbitrary polynomial of degree at most $n-1$ with $B(\rho_1)\neq 0$. Define a sequence of polynomials of degree $n-1$ by

$$G(0, t, B) = B(t), \quad G(\lambda + 1, t, B) = tG(\lambda, t, B) - \alpha_0(\lambda)P(t), \qquad \lambda = 0, 1, \cdots,$$

where $\alpha_0(\lambda)$ is the leading coefficient of $G(\lambda, t, B)$. From the polynomial $G(\lambda, t, B)\equiv G_1(\lambda, t, B)$, form the polynomial $G_p(\lambda, t, B)$ for any positive integer p by

$$G_p(\lambda, t, B) = \sum_{k=0}^{p-1-k} [-P]^{p-1-k} \frac{G^{(p-1-k)}(\lambda, t, B)}{(p-1-k)!} V_k(t),$$

where $V_k(t)$ is formed by

$$V_0(t) = 1, \quad V_k(t) = P'(t)V_{k-1}(t) - \frac{P(t)}{k} V'_{k-1}(t).$$

Define an iteration function for *fixed* p and λ by

$$\phi_p(\lambda, t, B) = t - P(t)\frac{G_{p-1}(\lambda, t, B)}{G_p(\lambda, t, B)}.$$

The global nature of the convergence is given by

THEOREM. *Let t_0 be an arbitrary point in the extended complex plane such that $t_0\neq\rho_2, \rho_3, \cdots, \rho_n$ and let $t_{i+1}=\phi_p(\lambda, t_i, B)$. Then for all sufficiently large but fixed λ, the sequence t_i is defined for all i and $t_i\to\rho_1$.*

REMARK. Since $\phi_p(\lambda, \rho_j, B) = \rho_j$, $j = 1, 2, \cdots, n$, we have an:

ALTERNATIVE FORMULATION. *Let t_0 be an arbitrary point in the extended complex plane and let $t_{i+1} = \phi_p(\lambda, t_i, B)$. Then for all sufficiently large but fixed λ the sequence t_i is defined for all i and $t_i \to \rho_j$ for some j.*

Observe that the sequence is defined for all i. This should be compared with a sequence generated by, say the Newton-Raphson iteration function where the sequence is not defined at the zeros of P'.

The asymptotic rate of convergence is given by

$$\lim_{t \to \rho_1} \frac{\phi_p(\lambda, t, B) - \rho_1}{(t - \rho_1)^p} = C_p(\lambda, B),$$

$$C_p(\lambda, B) = -\sum_{i=2}^{n} \frac{d_i \mu_i^{\lambda}}{(\rho_1 - \rho_i)^{p-1}},$$

where

$$d_i = \frac{B(\rho_i)}{B(\rho_1)} \frac{P'(\rho_1)}{P'(\rho_i)}, \qquad \mu_i = \frac{\rho_i}{\rho_1}, \qquad i = 2, 3, \cdots, n.$$

Hence the order of $\phi_p(\lambda, t, B)$ is p while the asymptotic error constant (Traub [1, p. 9]) is given by $C_p(\lambda, B)$. Observe that

$$\lim_{\lambda \to \infty} C_p(\lambda, B) = 0.$$

Results on the character of the convergence are readily available. An example is furnished by the following

THEOREM. *Let $|\rho_1| > |\rho_2| > |\rho_j|$, $j = 3, 4, \cdots, n$. Choose p and λ even and $B = P'$. Let λ be sufficiently large. Then if $\rho_1 > \rho_2$, $t_i \uparrow \rho_1$; if $\rho_1 < \rho_2$, $t_i \downarrow \rho_1$.*

The iteration functions $\phi_p(0, t, 1)$ are classical. Calculating $\lim_{\lambda \to \infty} \phi_p(\lambda, \infty, B)$ or $\lim_{p \to \infty} \phi_p(0, 0, 1)$ are classical noniterative methods for approximating the zeros of polynomials.

The proofs of these and additional results will appear elsewhere.

REFERENCE

1. J. F. Traub, *Iterative methods for the solution of equations*, Prentice-Hall, Englewood Cliffs, N. J., 1964.

BELL TELEPHONE LABORATORIES, INCORPORATED, MURRAY HILL, NEW JERSEY