A NEW LOCAL PROPERTY OF EMBEDDINGS

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It is known that the possible embeddings of a topological $n-1$ manifold M^{n-1} in the euclidean space E^n differ in the cases $n=3$ and $n>3$ in a curious way. A topological $n-1$ sphere can fail to be locally flat at an arbitrary finite number of points if $n=3$. For $n>3$ this cannot happen at a set consisting of a single point [2]. It is unresolved if an S^{n-1} in E^n can fail to be locally flat at a pair of points. In this note we introduce a new notion, described in detail below, called a *locally weakly flat* embedding and show that if a manifold M^{n-1} in E^n is locally flat at each point except possibly at the points of a finite set Y and if M^{n-1} is locally weakly flat at each point of Y, then M^{n-1} is in fact locally flat at every point. In the concluding paragraph an unsolved problem is posed.

Let $p \in M^k \subset E^n$, or more generally $M^k \subset M^n$. Suppose $\epsilon > 0$. Let B^n_ϵ be a ball of diameter less than ϵ whose interior contains p. For $0 < t \leqq \epsilon$ let B_t denote a ball whose interior contains p and is concentric to B^n_ϵ, i.e., regard B_t as a topological product $S^{n-1} \times [0, t]$ with $S^{n-1} \times [0]$ identified with p. For all t such that $\epsilon - t$ is sufficiently small we hypothesize that $\dot{B}_t \cap M$ is a $k-1$ sphere such that the pairs

$$(E^n, \dot{B}^n_t \cap M^k \times I^{n-k+1}) \approx (E^n, S^{k-1} \times I^{n-k+1})$$

are homeomorphic. If for a sequence of positive numbers $\epsilon_1, \epsilon_2, \cdots$ converging to zero, this condition holds, we describe the embedding by saying M^k is locally weakly flat at p. If this holds for all $p \in M^k$, M^k is locally weakly flat in M^n, denoted by LWF.

A comparison with other local properties of embeddings [3] shows that LF = LU$\Rightarrow$LWF$\Rightarrow$LSPU$\Rightarrow$LPU.

For $n=3$, $k=2$ these implications may be reversed [4]. There are examples, for $n=3$, that show that at a single point, local peripheral unknottedness, or local weakly flatness does not imply local flatness [5].

For $n=3$, $k=1$, LU and LPU are entirely independent. In this paper attention is restricted to $k=n-1$.

THEOREM. *Let $M^{n-1} \subset E^n$ be a closed $n-1$ manifold that is locally*

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flat at each point except possibly at the points of a finite set Y. Suppose that M^{n-1} is LWF at each point. Then M^{n-1} is locally flat at each point.

The proof rests on an adaptation of a theorem of M. Brown's to what I refer to as the "Turning Lemma" for annuli. The same idea can be used to establish a "Union Lemma" for $n-1$ disks in E^n.

Notations. In order to ease our descriptions we define once and for all the meaning of

(1) nice k-disk in S^k, denoted by D^k;
(2) nice k-disk in E^{k+1}, denoted by D^k;
(3) nice k-annulus in S^k, denoted by A^k;
(4) nice k-annulus in E^{k+1}, denoted by A^k.

By (1) we mean the boundary ∂D^k of D^k has a shell neighborhood in S^k. By (2) we mean that D^k is the image of an equatorial plane section under some homeomorphism of a standard $k+1$ ball into E^{k+1}. By (3) we mean each boundary component of ∂A^k, the boundary of A^k, has a shell neighborhood in S^k. By (4) we mean A^k is the image of an equatorial plane section under some heomormophism of a standard $I^2 \times S^{k-1}$ into E^{k+1}.

Some recent results needed for the proof. 1. Let h be a homeomorphic embedding of $S^n \times [-1, 1]$ into S^{n+1}, where $[a, b]$ denotes the closed real number interval $a \leqq t \leqq b$. Then the closure of either complementary domain of $h(S^n \times [0])$ is an $(n+1)$-cell (Theorem 5 of *A proof of the generalized Schoenflies theorem*, M. Brown).

2. Let B be a subset of a metric space X. Suppose $B = U_1 \cup U_2$, where U_1, U_2 are open in B and $U_1 \cap U_2 \neq \square$. If both U_1, U_2 are collared in X, then B is collared in X. If B is an orientable bounded manifold of dim n in E^{n+1}, and B is collared on each "side," B is bicollared at each point of $B \backslash \partial B$. (Lemma 4 of *Locally flat embeddings of topological manifolds*, M. Brown [1]).

3. Let D_1 and D_2 be topological n-disks in E^{n+1}. Suppose each of D_1 and D_2 is nice (see above under Notations). Let $D_1 \cap D_2 = \partial D_1 \cap \partial D_2 = S^{n-1}$. Suppose S^{n-1} lies in the interior of a nice annulus A that is a subset of $D_1 \cup D_2$.[2] Then $S = D_1 \cup D_2$ is nice.

3′. Let $\tilde{D}$ and $\tilde{A}$ be respectively a nice n-disk, a nice n-annulus in E^{n+1}. Suppose $\tilde{D} \cup \tilde{A}$ is a disk. Further $\partial \tilde{D}$ lies in Int $\tilde{A}$. Then $\tilde{D} \cup \tilde{A}$ is a nice disk in E^{n+1}. The proofs of 3 and 3′ are so similar to that of 3″ we omit them.

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[2] The symbol "int" occurs in two senses. The meaning will be clear since in one case it means the bounded component of the complement of a set and in the other case it refers to the points not on the combinatorial boundary of some manifold with boundary.

3''. THE TURNING LEMMA. *Let F be a homeomorphism, F: $S^{n-1}\times I^2 \to E^{n+1}$. Let I_1 and J_1 be intervals lying in the interior of I^2 such that $I_1 \cap J_1 = \{0\}$, an endpoint of each of them. Suppose*

(i) $F|S^{n-1}\times I_1 = A_1$, $F|S^{n-1}\times J_1 = A_2$, *and*

(ii) $F|S^{n-1}\times \{0\} = S^{n-1}$.

Then $A_1 \cup A_2$ is nice in E^{n+1}.

To put it another way, whenever two n-annuli A_1 and A_2 are nice in E^{n+1} and their common part is a component S_{12} of the boundary of each of them, and if F satisfies the *consistency* conditions (i) and (ii) above, then $A_1 \cup A_2$ is nice.

PROOF. Let g be a homeomorphism of I^2 on I^2 so that $I_1 \cup J_1$ is carried onto $I_1 \cup J_1$ carrying $\{0\}$ into an inner point of J_1, leaving the other endpoints fixed,[3] and also leaving the points of $S' = \partial I^2$ fixed. Then

$$G(x, y) = F(x, g(y))$$

defines a homeomorphism of $S^{n-1}\times I^2$ onto $F(S^{n-1}\times I^2)$ and A_1 onto $\tilde{A}_1$ ⟨say⟩. Then $\text{Int}(A_1 \cup A_2) = \text{Int } A_2 \cup \text{Int } \tilde{A}_1$ and $\text{Int } A_2 \cap \text{Int } \tilde{A}_1$ is open and non-null. Then if $B = (\text{Int } A_2) \cup (\text{Int } \tilde{A}_1)$, B is collared,[4] and, in fact bi-collared. Hence $A_1 \cup A_2$ is nice in E^{n+1}.

PROOF OF THE THEOREM. Let p be a point of Y and ϵ sufficiently small that $S(p, \epsilon) \cap (Y \backslash p) = \square$ (the empty set).

Let $B_1, B_2, \cdots$ be a sequence of balls with diameter approaching zero that are "concentric" about p, each of which meets M nicely, as guaranteed by the condition $(E^n, \dot{B}_i^n \cap M \times I^2) \approx (E^n, S^{n-2}\times I^2)$. The spheres $\dot{B}_1, \dot{B}_2, \cdots$ may be taken disjoint. Let $\dot{B}_i$ and $\dot{B}_{i+1}$ determine an annulus A_i on M. Since $\dot{B}_i \cap M$ is nice in M, a homeomorphism of B_i onto itself moving points an arbitrarily small amount may be defined to insure A_i is an annulus. The boundary components of A_i are denoted by S_i and S_{i+1}. Let B_i be decomposed by S_i into two components C_i^N and C_i^S, whose closures are closed $n-1$ disks and the notation is chosen so that $C_1^N, C_2^N, \cdots$ all lie on the same side of $E^n \backslash M$. Since S^{n-2} is nicely embedded in E^n, it is clear that the consistency conditions required in the hypotheses of 3' above hold for S_i relative to A_i and C_i^N. Hence $A_i \cup C_i^N$ is a nice disk F_i. Since S_{i+1} is nice relative to C_{i+1}^N and A_i, $C_{i+1}^N \cup A_i$ is a nice $n-1$ disk G_i. The conditions of 3 (above) are fulfilled so that $F_i \cup G_i$ is a flat $n-1$ sphere. By passing $n-1$ planes parallel to the base of an n-simplex that converge to the a

[3] Such a homeomorphism is easily found via the plane Schoenflies theorem.

[4] This is the content of Lemma 4 of [1].

vertex, one may slice the n-simplex into a sequence of nice n-cells $\sigma_1^n, \cdots$ with diameters approaching zero. By mapping each σ_i^n to $F_i \cup G_i$ so that the consecutive functions agree on the common face of σ_i^n and σ_{i+1}^n, the manifold M is seen to have a collar at p relative to the complementary domain determined by $C_1^N, \cdots$. A similar construction of the other side of M shows that M is in fact locally bi-collared at p.

By noting that the set Y of M consisting of points where M fails to be locally flat is closed, it is easy to extend the above theorem to the case cardinal of $Y \leqq \aleph_0$.

Added in proof. COROLLARY. *If S is an* $n-1$ *sphere that is locally flat except possibly at two points p and q and if S is LWF at either p or q, then S is flat.*[5]

A question we have been unable to resolve is contained in the following.

PROBLEM. If M^{n-1} is LWF, is it LF in M^n? The result is known to be true for $n=3$.

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[5] One may define a concept of M^{n-1} being LWF with respect to the complementary domain A (or the other complementary domain B) and derive a similar result.