ESSENTIALLY POSITIVE SYSTEMS OF LINEAR DIFFERENTIAL EQUATIONS

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Consider an essentially positive [2] system of linear DE's, that is, any system of the form

$$dx_i/dt = \sum q_{ij}(t)x_j, \qquad q_{ij}(t) > 0 \text{ if } i \neq j. \tag{1}$$

This maps the positive hyperoctant C of real $(x_1, \cdots, x_n)$-space into itself. The effect of (1) in a small increment of time dt is, using formulas of Ostrowski [3], to map C into an *interior* polyhedral cone whose projective diameter is given for $P = \exp[Q(t)dt] = I + Q(t)dt + \cdots$ by

$$\ln\{\sup_{i,j,k,l}(p_{ki}p_{lj}/p_{kj}p_{li})\}. \tag{2}$$

This is maximized asymptotically (as $dt \downarrow 0$) by setting $k=i$, $l=j$, so that the numerator approaches 1. Hence the projective diameter of $e^{Q(t)dt}(C)$ is, asymptotically,

$$\Delta = -\ln\{\inf_{i\neq j}[(q_{ij}q_{ji})dt^2]\}, \qquad q_{ij} = q_{ij}(t). \tag{3}$$

By a basic theorem of [1], all projective distances in C are therefore contracted by a factor at most

$$\tanh(\Delta/4) = (1 - e^{-\Delta/2})/(1 + e^{-\Delta/2}) = 1 - \psi(t)dt, \quad \text{where } \psi(t) = 2[\inf_{i\neq j}q_{ij}(t)q_{ji}(t)]^{1/2}. \tag{4}$$

This proves the following basic result.

LEMMA. *For any essentially positive system* (1) *of linear* DE'*s, all projective distances in C are contracted by an asymptotic factor at most* $1-\psi(t)dt$ *in the time interval* $(t, t+dt)$, *where* $\psi(t)$ *is given by* (4).

Integrating with respect to t, we deduce the

THEOREM. *For any essentially positive system* (1) *of linear* DE'*s, let* $\theta(\mathbf{x}(t), \mathbf{y}(t))$ *denote the projective distance in C between two solutions of* (1) *which are positive on* $[0, \infty)$. *Then*

$$\theta(\mathbf{x}(t), \mathbf{y}(t)) \leqq \theta(\mathbf{x}(0), \mathbf{y}(0)) \exp\left[-\int_0^t \psi(s)ds\right], \qquad t > 0. \tag{5}$$

For example, consider the interesting case $d^2x/dt^2 = p(t)x$, $p(t) > 0$. Then

$$Q(t) = \begin{pmatrix} 0 & 1 \\ p(t) & 0 \end{pmatrix}$$

has only one pair of off-diagonal entries, and $\psi(t) = 2\sqrt{(p(t))}$. Hence we have the

COROLLARY. *Let $x(t)$ and $y(t)$ be any two solutions of $d^2x/dt^2 = p(t)x$ with $p(t) > 0$, $x(0) > 0$, $x'(0) > 0$, $y(0) > 0$, $y'(0) > 0$. Then*

$$\left|\ln\left[\frac{x(t)y'(t)}{x'(t)y(t)}\right]\right| \leqq \left|\ln\left[\frac{x(0)y'(0)}{x'(0)y(0)}\right]\right| \exp\left[-2\int_0^t (p(s))ds\right]. \tag{6}$$

The preceding result is helpful for proving the convergence of all positive solutions of an essentially positive system (1) to a unique limiting asymptotic behavior as $t \to \infty$.

To treat similarly $d^3x/dt^3 = p(t)x$ with $p(t) > 0$, however, the preceding technique must be modified.

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