SOLVABILITY OF THE FIRST COUSIN PROBLEM AND VANISHING OF HIGHER COHOMOLOGY GROUPS FOR DOMAINS WHICH ARE NOT DOMAINS OF HOLOMORPHY[1]

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This work is a sequel to [1]: In [1] we considered the vanishing of the first cohomology groups with coefficients in $\mathcal{O}$, $\mathcal{O}^*$ for sets $X\backslash A$ whereas in the present work we consider the same question for higher cohomology; at the same time we obtain some additional results for the first Cousin problem. As in [1] we take $n\geqq 3$.

Scheja [3] proved that if X is an open set in $\mathbf{C}^n$ and A is an analytic closed subset of X of dimension $\leqq n-q-2$, then the natural homomorphism

$$H^q(X, \mathcal{O}) \to H^q(X\backslash A, \mathcal{O}) \tag{1}$$

is bijective. We shall prove:

THEOREM 1. *Let A be a closed bounded generalized polydisc in an open set X of $\mathbf{C}^n$. Then the natural homomorphism* (1) *is bijective for any* $1\leqq q\leqq n-2$.

PROOF. Set $A=L_1\times \cdots \times L_n$ and let $K=K_1\times \cdots \times K_n$ be an open generalized polydisc with $A \subset K \subset \overline{K} \subset X$. Set $L' = L_2 \times \cdots \times L_n$, $K'=K_2\times \cdots \times K_n$, $G_0=(K_1\backslash L_1)\times K'$, $G_1=K_1\times(K'\backslash L')$, $G=G_0\cup G_1$. By a straightforward generalization of [3, Hilfsatz] one gets $H^q(G, \mathcal{O})=0$. We now introduce a covering $U=\{U_i\}$ of $X\backslash A$ where all the U_i are domains with $H^q(U_i, \mathcal{O})=0$ and precisely $q+1$ of them, say $U_{i_0}, \cdots, U_{i_q}$, coincide with G. By Leray's theorem [2], the canonical homomorphism

$$H^q(N(U), \mathcal{O}) \to H^q(X\backslash A, \mathcal{O}) \tag{2}$$

(where $N(U)$ is the nerve of U) is bijective.

We next introduce a covering $U'=\{U_i'\}$ of X where $U'_{i_0}= \cdots = U'_{i_q}=K_1\times K'$ and $U_i' = U_i$ for all other indices i. Again, the canonical map

$$H^q(N(U'), \mathcal{O}) \to H^q(X, \mathcal{O}) \tag{3}$$

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is bijective. We shall now construct a map

$$H^q(N(U), \mathcal{O}) \to H^q(N(U'), \mathcal{O}). \tag{4}$$

Let $f \in H^q(N(U), \mathcal{O})$. We may view it as a q-cocycle. Let $f_{i_0 \cdots i_q}$ be the section of f on $U_{i_0} \cap \cdots \cap U_{i_q} = G$. The proof of Lemma 3 in [1] can be extended to show that $f_{i_0 \cdots i_q}$ can be continued analytically to $K_1 \times K'$. The continued function $f'_{i_0 \cdots i_q}$ thus obtained is defined on $U'_{i_0} \cap \cdots \cap U'_{i_q}$. We now define $f'_{j_0 \cdots j_q}$ for any set of distinct indices $\{j_0, \cdots, j_q\}$ which does not coincide with the set $\{i_0, \cdots, i_q\}$. Since among the j_h's there is at least one index, say i, with $i \neq i_k$ for all $0 \leqq k \leqq q$, and, consequently, $U'_i = U_i \subset X \backslash A$, we have $U'_i \cap (K_1 \times K') = U'_i \cap G$. Hence $U'_{j_0} \cap \cdots \cap U'_{j_q} = U_{j_0} \cap \cdots \cap U_{j_q}$ and we can take $f'_{j_0 \cdots j_q} = f_{j_0 \cdots j_q}$.

We have thus defined a q-cochain f' on $N(U')$. f' is cocycle. Indeed, observing that $U'_{j_0} \cap \cdots \cap U'_{j_{q+1}}$ coincides with $U_{j_0} \cap \cdots \cap U_{j_{q+1}}$ if all the j_k are distinct from each other, and that the analytic function $f'_{j_0 \cdots \hat{j}_\nu \cdots j_{q+1}}$ restricted to either of these sets coincides with $f_{j_0 \cdots \hat{j}_\nu \cdots j_{q+1}}$, the equation $\delta f = 0$ implies $\delta f' = 0$.

We next show that if $f = \delta g$ then there is a $(q-1)$-chain g' with $\delta g' = f'$. If (a) $\{j_0, \cdots, j_{q-1}\} \subset \{i_0, \cdots, i_q\}$ then we take $g'_{j_0 \cdots j_{q-1}}$ to be the analytic continuation of $g_{j_0 \cdots j_{q-1}}$ to $U'_{j_0} \cap \cdots \cap U'_{j_{q-1}}$, whereas if (a) does not hold then $U'_{j_0} \cap \cdots \cap U'_{j_{q-1}} = U_{j_0} \cap \cdots \cap U_{j_{q-1}}$ and we take $g'_{j_0 \cdots j_{q-1}} = g_{j_0 \cdots j_{q-1}}$. With g' thus constructed, the relation $\delta g' = f'$ over $U'_{j_0} \cap \cdots \cap U'_{j_q}$ in case (b) $\{j_0, \cdots, j_q\} = \{i_0, \cdots, i_q\}$ follows from the relation $\delta g = f$ over $U_{j_0} \cap \cdots \cap U_{j_q}$ by analytic continuation, whereas in case (b) does not hold it coincides with the relation $\delta g = f$ over $U_{j_0} \cap \cdots \cap U_{j_q}$.

We have thus shown that the map $f \to f'$ defines a homomorphism (4). This map is surjective since, given f', its restriction f to $N(U)$ is mapped into f' by the above map. It is also injective since if $f' = \delta g'$ for some $(q-1)$-cochain g' over $N(U')$, then the restriction g of g' to $N(U)$ clearly satisfies $f = \delta g$. Noting finally that the map $f \to f'$ is the inverse of the restriction map, and combining (2)–(4), (1) follows.

COROLLARY. *If $H^q(X, \mathcal{O}) = 0$ then $H^q(X \backslash A, \mathcal{O}) = 0$. In particular, if X is Cousin* I *then $X \backslash A$ is Cousin* I.

THEOREM 2. *Let A, B be two closed bounded subsets of an open set $X \subset \mathbf{C}^n$ and let P be a closed generalized polydisc with $A \subset \operatorname{int} P \subset P \subset \operatorname{int} B$. If, for some $1 \leqq q \leqq n-2$, the natural homomorphism*

$$H^q(X \backslash A, \mathcal{O}) \to H^q(X \backslash B, \mathcal{O}) \tag{5}$$

is injection, then there exists a homomorphism $\lambda: H^q(X \backslash A, \mathcal{O}) \to H^q(X, \mathcal{O})$

such that $\pi\lambda =$ *identity, where* π *is the map* (1) (*and, consequently,* π *is surjective*); *in particular, if* $H^q(X, \mathcal{O}) = 0$ *then* $H^q(X\backslash A, \mathcal{O}) = 0$.

PROOF. Take coverings U^1, U^2, U^3, U^4 of X, $X\backslash A$, $X\backslash P$, $X\backslash B$ respectively whose open sets are domains of holomorphy and such that the sets of U^i ($i = 2, 3, 4$) are among the sets of U^{i-1}. Given $f_2 \in H^q(N(U^2), \mathcal{O})$ there corresponds to it (by restriction) a unique element f_4 in $H^q(N(U^4), \mathcal{O})$ and a unique element f_3 in $H^q(N(U^3), \mathcal{O})$; f_4 is the restriction of f_3. By Theorem 1 there exists an $f_1 \in H^q(N(U^1), \mathcal{O})$ whose restriction to $N(U^3)$ is f_3. Hence the restriction of f_1 to $N(U_4)$ is f_4. Since f_1 and f_2 have the same restriction on $N(U^4)$, the injectivity of (5) implies that the restriction of f_1 to $N(U^2)$ is f_2. Thus the map $f_2 \to f_1$ is an inverse of the restriction map $H^q(N(U^1), \mathcal{O}) \to H^q(N(U^2), \mathcal{O})$. The assertion of the theorem now follows with λ being the image of the homomorphism $f_2 \to f_1$ under the canonical map corresponding to $H^q(N(U^2), \mathcal{O}) \to H^q(X\backslash A, \mathcal{O})$, $H^q(N(U^1), \mathcal{O}) \to H^q(X, \mathcal{O})$.

GENERALIZATIONS. By successive applications of Theorem 1 we get:

(1) If $A_1, \cdots, A_m$ are closed bounded generalized polydiscs such that $A_j \cap A_k = \emptyset$ if $j \neq k$, then the natural map

$$H^q(X, \mathcal{O}) \to H^q\left(X \Big\backslash \left(\bigcup_{i=1}^m A_i\right), \mathcal{O}\right)$$

is bijective.

(2) Theorem 1 extends to the case where X is an open set on a complex manifold provided A is contained in one coordinate patch and its image in $\mathbf{C}^n$ is a generalized polydisc. Theorem 2 and (1) have similar extensions.

By slightly modifying the proof of Theorem 1 we obtain:

(3) If $X = X_1 \times K_{p+1} \times \cdots \times K_n$, $A = A_1 \times K_{p+1} \times \cdots \times K_n$ where X_1 is any open set of $\mathbf{C}^p$ and K_j is an open set in the z_j-plane, then the homomorphism (1) is bijective if $1 \leq q \leq p-2$.

(4) If A in Theorem 1 is convex, then (see [1]) $H^q(G, \mathcal{O}^*) = 0$. By modifying the proof of Theorem 1 we find that the natural homomorphism

$$H^q(X, \mathcal{O}^*) \to H^q(X\backslash A, \mathcal{O}^*)$$

is bijective. The analogs of Theorem 2 and (1)–(3) are also valid.

We shall now give a different approach to proving results similar to Theorem 1. Since this approach does not yield a result as general as Theorem 1, we shall only sketch it. Let $X = K_1 \times \cdots \times K_n$, $A = L_1 \times \cdots \times L_n$ be generalized polydiscs. We say that the condition (A_m) holds if for each $j = 1, \cdots, m$ either (a) K_1 is the whole plane $\mathbf{C}$ and then L_j is an arbitrary closed bounded subset of K_j, or (b) $K_j = \mathbf{C}$ and then L_j consists of a finite number of points. The L_j for $j = m+1, \cdots, n$ are arbitrary closed subsets of K_j.

THEOREM 3. *If* (A_m) *holds for some* $2 \leqq m \leqq n$ *then* $H^q(X \backslash A, \mathfrak{O}) = 0$ *for* $1 \leqq q \leqq \min(m-1, n-2)$. *The relations* $H^{n-1}(X \backslash A, \mathfrak{O}) \neq 0$, $H^q(X \backslash A, \mathfrak{O}) = 0$ *for* $q \geqq n$ *are valid under the assumption* (A_0).

PROOF. Setting $\Delta_j = K_1 \times \cdots \times K_{j-1} \times (K_j \backslash L_j) \times K_{j+1} \times \cdots \times K_n$ and noting that $H^q(\Delta_j, \mathfrak{O}) = 0$ for $q \geqq 1$, it suffices to consider $H^q((U), \mathfrak{O})$, where $U = \{\Delta_1, \cdots, \Delta_n\}$. We consider only the case $1 \leqq q \leqq n-2$. Denote by $I_{j_1 \cdots j_k}(h)$ the Cauchy integral of h with the ith contour being ∂K_i if $i \neq j_p$ for all p, and ∂L_i if $i = j_p$ for some p. (Actually one should replace ∂K_m, ∂L_m by smooth $\partial K_{m,\epsilon}$, $\partial L_{m,\epsilon}$ which approximate ∂K_m, ∂L_m.) Then we can represent each component $f_{i_0 \cdots i_q}$ of a q-cochain f by

$$f_{i_0 \cdots i_q} = \sum_{k=0}^{q+1} \sum_{0; j_1 < \cdots < j_k}^{q} I_{i_{j_1} \cdots i_{j_k}}(f_{i_0 \cdots i_q}). \tag{6}$$

LEMMA 1. *Consider a domain* $D = K \backslash L$ *in the complex plane, where* K *is the whole plane and* L *is any closed bounded set with* C^1 *boundary* ∂L. *Let* $\phi(z)$ *be any analytic function in* D *and let* $\psi(z)$ *be any continuous function on* ∂L *such that*

$$\int_{|\zeta|=R} \frac{\phi(\zeta)}{\zeta - z} d\zeta + \int_{\partial L} \frac{\psi(\zeta)}{\zeta - z} d\zeta = 0 \quad in \quad D \cap \{z;\ |z| < R\}$$

for all R *sufficiently large. Then, for all* R *sufficiently large,*

$$\int_{|\zeta|=R} \frac{\phi(\zeta)}{\zeta - z} d\zeta = \int_{\partial L} \frac{\psi(\zeta)}{\psi - z} d\zeta = 0 \quad in \quad D \cap \{z;\ |z| < R\}.$$

A similar result holds in case K is a bounded set with C^1 boundary and L consists of a finite number of points. Using these results, the condition $\delta f = 0$ implies the following system of equations:

If $i_0 < \cdots < i_h \leqq m < i_{h+1} < \cdots < i_{q+1}$ for some $0 \leqq h \leqq q+1$, and if $i_{j_1} < \cdots < i_{j_k} \leqq m$ for some $0 \leqq k \leqq h$, then

$$\sum_{p=0}^{q-h+1} \sum_{h+1; \lambda_1 < \cdots < \lambda_p}^{q+1} I_{i_{j_1} \cdots i_{j_k} i_{\lambda_1} \cdots i_{\lambda_p}} \left(\sum_{\nu=0}^{q+1} (-1)^\nu f_{i_0 \cdots \hat{i}_\nu \cdots i_q} \right) = 0, \tag{7}$$

where in the third summation $\nu \neq j_1, \cdots, \nu \neq j_k$ and $\nu \neq \lambda_1, \cdots, \nu \neq \lambda_p$.

To find g satisfying $\delta g = f$, we try to represent $g_{i_0 \cdots i_{q-1}}$ analogously to (6), and then the relation $\delta g = f$ is a consequence of the following system of equations:

If $i_0 < \cdots < i_{h-1} \leqq m < i_h < \cdots < i_q$ for some $0 \leqq h-1 \leqq q$, and if $i_{j_1} < \cdots < i_{j_k} \leqq m$ for some $0 \leqq k \leqq h-1$, then

$$\sum_{p=0}^{q-h+1} \sum_{h;\lambda_1<\cdots<\lambda_p}^{q} I_{i_{j_1}\cdots i_{j_k} i_{\lambda_1}\cdots i_{\lambda_p}}\left(\sum_{\nu=0}^{q}(-1)^\nu g_{i_0\cdots\hat{i}_\nu\cdots i_q}\right) - \sum_{p=0}^{q-h+1} \sum_{h;\lambda_1<\cdots<\lambda_p}^{q} I_{i_{j_1}\cdots i_{j_k} i_{\lambda_1}\cdots i_{\lambda_p}}(f_{i_0\cdots i_q}) = 0, \tag{8}$$

where in the third summation of the first term $\nu \neq j_1, \cdots, \nu \neq j_k$ and $\nu \neq \lambda_1, \cdots, \nu = \lambda_p$.

Using (7) we can solve (8) as follows: If $i_0 > 1$, or if $i_0 = 1$, $i_{j_1} > 1$ then $g_{i_0\cdots i_{q-1}} = f_{1 i_0 \cdots i_{q-1}}$. If $i_0 = i_{j_1} = 1$ and if $i_1 > 2$ or $i_1 = 2$, $i_{j_2} > 2$ then $g_{i_0\cdots i_{q-1}} = f_{2 i_0\cdots i_{q-1}}$. We proceed in this manner and finally define, in case $i_0 = i_{j_1} = 1, \cdots, i_{k-1} = i_{j_k} = k$, $g_{i_0\cdots i_{q-1}} = f_{k+1, i_0 \cdots i_{q-1}}$.

This method extends also to the situations described in (1), (3) above.

Added in proof. The relation $H^{p-2}(X\backslash A, \mathcal{O}) \neq 0$ holds if in (3) X_1 and A_1 are both generalized polydiscs. Taking $\Omega_m = X_m \backslash A_m$ where X_m, A_m are generalized polydiscs with $X_m \searrow \overline{X}$, $A_m \nearrow A$ one derives, for fixed $1 \leqq q \leqq n-2$, examples of domains Ω_m with $\Omega_{m-1} \supset \overline{\Omega}_m$, such that $H^r(\Omega_m, \mathcal{O}) = 0$ for $1 \leqq r \leqq n-2$ but $H^q(\Omega, \mathcal{O}) \neq 0$ where $\Omega = \operatorname{int}(\lim \Omega_m)$.

By Dolbeault's theorem, $H^q(\Omega, \mathcal{O}) = 0$ if and only if for any $C^\infty(\Omega)$ form f of bidegree $(0, q)$ with $\bar\partial f = 0$ there is a $C^\infty(\Omega)$ form u with $\bar\partial u = f$. By modifying the proof in [**2**, p. 29] we find: If for some $q > 1$, $\overline{\Omega}_m \subset \Omega_{m+1}$, $\Omega = \lim \Omega_m$, $H^r(\Omega_m, \mathcal{O}) = 0$ for $r = q-1, q$, then $H^q(\Omega, \mathcal{O}) = 0$. Also if $H^1(\Omega_m, \mathcal{O}) = 0$ and if for any u holomorphic in Ω_m and $\epsilon > 0$ there is a v holomorphic in Ω_{m+1} with $|u - v| < \epsilon$ in Ω_{m-1}, then $H^1(\Omega, \mathcal{O}) = 0$; this can be applied to $\Omega_m = X_m \backslash A_m$ as in [**1**, Theorem 3].

References

1. A. Friedman, *On the Cousin problems*, Bull. Amer. Math. Soc. **71** (1965), 737–741.

2. R. C. Gunning and H. Rossi, *Analytic functions of several complex variables*, Prentice-Hall, 1965.

3. G. Scheja, *Riemannsche Helbarkeitssätze für Cohomologieklassen*, Math. Ann. **144** (1961), 345–360.

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