ON THE COUSIN PROBLEMS[1]

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It is well known that if Ω is a domain of holomorphy in C^n then it is a Cousin I domain; it is also a Cousin II domain if and only if $H^2(\Omega, Z)=0$. In this work we prove that some general classes of domains which are not domains of holomorphy are both Cousin I and Cousin II domains. Recall that Ω is Cousin I (II) if and only if $H^1(\Omega, \mathcal{O})=0$ ($H^1(\Omega, \mathcal{O}^*)=0$) where $\mathcal{O}$ is the sheaf of germs of holomorphic functions under addition and $\mathcal{O}^*$ is the sheaf of germs of nowhere-zero holomorphic functions under multiplication. If $H^1(\Omega, Z)=0$ then "Ω Cousin II" implies "Ω Cousin I" and if $H^2(\Omega, Z)=0$ then "Ω Cousin I" implies "Ω Cousin II."

In what follows we take $n\geqq 3$ since, for $n=2$, Ω is Cousin I if and only if Ω is a domain of holomorphy [**1**].

DEFINITIONS. An open relatively compact set A in a complex manifold X is called *q-convex* if $A=\{z;\ z\in A_0,\ \phi(z)<0\}$ where A_0 is a neighborhood of $\bar{A}$, ϕ is twice continuously differentiable in A_0, grad $\phi\neq 0$ on ∂A, and the Levi form on ∂A has at least $n-q+1$ positive eigenvalues. If A and B are q-convex, $B\subset A$, and if there exists a function $\phi(z, t)$ ($z\in A_0$, $0\leqq t\leqq 1$) twice continuously differentiable in z such that the sets $D_t=\{z;\ z\in A_0,\ \phi(z, t)<0\}$ are q-convex and lie in A_0 and $D_0=A$, $D_1=B$, then we say that A and B are *q-convex homotopic*. Example: if A, B are strictly convex then they are 1-convex homotopic.

Let K_1, L_1 be open convex sets in the z_1-plane, $0\in L_1$, $\bar{L}_1\subset K_1$, and set $A_1=K_1\backslash\bar{L}_1$. Let $K'=K_2\times\cdots\times K_n$, $L'=L_2\times\cdots\times L_n$ be open convex generalized polydiscs (K_j, L_j lie in the z_j-plane) with $0\in L'$, $\bar{L}'\subset K'$. All the previous sets are taken to be bounded. Set $G_0=A_1\times K'$, $G_1=K_1\times(K'\backslash\bar{L}')$, $G=G_0\cup G_1$.

LEMMA 1. *G is both Cousin* I *and Cousin* II.

The proof that G is Cousin I is a straightforward generalization of the proof of [**7**, Hilfsatz]. Thus, it remains to show that $H^2(G, Z)=0$.

LEMMA 2. $H^r(G, Z)=0$ *for* $0<r\leqq 2n$.

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PROOF.[2] Moving the points of $\bar{K}_1\backslash L_1$ and of $\bar{K}'\backslash L'$ inward along the rays from 0 we get a strong deformation retract of G into a set homeomorphic to $N=S_a^1\times D_b^{m+1}\cup D_a^2\times S_b^m$ where $m=2n-1$, S_a^p and S_b^p are unit p-spheres, and D_a^p, D_b^p are unit p-balls. The join $S^1 \circ S^m$ can be described by $\mathbf{x}\cos t+\mathbf{y}\sin t$, $0\leqq t\leqq \pi/2$, where $\mathbf{x}\in S^1$, $\mathbf{y}\in S^m$. Introduce the function $(\vartheta, \phi, t)\to((1, \vartheta), (4t/\pi, \phi))$ which maps homeomorphically the set J_- in $S^1 \circ S^m$ corresponding to $0\leqq t\leqq \pi/4$ onto $S_a^1\times D_b^{m+1}$, where ϑ, ϕ are the angular coordinates of $\mathbf{x}$, $\mathbf{y}$. Similarly we introduce a map of J_+ (for which $\pi/4\leqq t\leqq\pi/2$) onto $D_a^2\times S_b^m$. Thus N is homeomorphic to $S^1\circ S^m$, and since the latter is known to be homeomorphic to S^{m+2}, the result follows.

LEMMA 3. *The envelope of holomorphy of G contains $K_1\times K'$.*

PROOF. Given f holomorphic in G, for any $\zeta\in A_1\times K'$ we represent $f(\zeta)$ by Cauchy's formula, where the z_1-contour is composed of one part lying near ∂K_1 and another "inner" part, say J, lying near ∂L_1, and where the z_j-contour, for $j\geqq 2$, is in $K_j\backslash\bar{L}_j$. Now notice that the integral over J vanishes.

LEMMA 4. *Let X be an open set in a complex manifold and let A, B be subsets of X n-convex homotopic, and $B\subset A$. Then any two Cousin* II *data in $X\backslash\bar{B}$ which are equivalent in $X\backslash\bar{A}$ are also equivalent in $X\backslash\bar{B}$.*

The proof is a rather obvious extension of [5, Satz 1, A1, A2] provided one employs a theorem of Lewy [4] (see also [3]) concerning local analytic continuation across the boundary of each ∂D_t.

LEMMA 5. *Let X be an open set in $\mathbf{C}^n$ and let $L=L_1\times\cdots\times L_n$ be a generalized polydisc which is open, convex and bounded, and $\bar{L}\subset X$. Then any Cousin* II *data (g_P, U_P) in $X\backslash\bar{L}$ can be continued into X.*

PROOF. Let $K=K_1\times\cdots\times K_n$ be an open convex bounded generalized polydisc with $\bar{L}\subset K$, $\bar{K}\subset X$ and introduce G as in Lemma 1. Clearly $G\subset X\backslash\bar{L}$. Since G is Cousin II, there exists an f holomorphic in G such that (f, G) is equivalent (in G) to the given Cousin data. Continue f to $K_1\times K'$ (by Lemma 3). For each P in $(K_1\times K')\cap G$ we take the germ f_P of f in the neighborhood G of P. For P in $(K_1\times K')\backslash G$ we take a sufficiently small neighborhood V_P of P such that its intersection with $X\backslash\bar{L}$ lies in G, and then take f_P to be the germ of the continuation of f. We have thus continued the Cousin data into X.

LEMMA 6. *Lemma* 5 *remains true if L is any open, strictly convex and bounded set with C^2 boundary, and $\bar{L}\subset X$.*

[2] I am indebted to Daniel Kahn for the proofs of this lemma and of Lemma 7.

PROOF. Let $R\in\partial L$. We first wish to continue the data to a neighborhood W of R. Assume that $\text{Re}(z_1)=0$ is the tangent hyperplane to L at R, that L lies in $\text{Re}(z_1)<0$, and that R is at the origin.

We apply a modified version of Lemmas 1–3 where G is defined differently, namely, $A_1=\{z_1;\ 0<\text{Re}(z_1)<d_0,\ |\text{Im}(z_1)|<\alpha\}$, $K_1=\{z_1,\ -d<\text{Re}(z_1)<d_0,\ |\text{Im}(z_1)|<\alpha\}$, and follow the argument of Lemma 5. We then need to show that the data obtained by the continuation of f agree with the given data in $(X\backslash\bar{L})\cap W$. This is done by extending the argument 3 of [5, p. 345]. However that argument is erroneous (since the existence of a smallest t^* is not justified). Instead we construct a family of C^2 hypersurfaces $S(t)$ $(1\leqq t\leqq 2)$ with boundary in $(K_1\backslash A_1)\times(K'\backslash\bar{L}')$ such that $S(t)$, at each of its points, is convex in at least one tangential direction (in fact, we can take it convex in $2n-1$ independent directions), $S(t)$ lies outside $\bar{L}$ if $t>1$, $S(1)\supset(X\backslash\bar{L})\cap W$, and $S(2)\subset G$. Then, by the proof of Lemma 4, we show that the set of t's such that at all the points of $S(\tau)$ $(t<\tau\leqq 2)$ the two sets of data are equivalent, is both open and closed. Having continued the data to W, the argument C of [5, p. 343], combined with Lemma 4, completes the proof.

DEFINITION. An n-convex subset A of $X\subset\mathbf{C}^n$ is said to have the property (P) if it is n-convex homotopic to a set $B\supset A$ $(\bar{B}\subset X)$, and if there exists a convex set L such that $\bar{A}\subset L\subset\bar{L}\subset B$.

THEOREM 1. *Let X be an open set in $\mathbf{C}^n$ and let A be an n-convex subset of X satisfying the property* (P). *Then any Cousin* II *data in $X\backslash\bar{A}$ can be continued into X.*

PROOF. Consider the given data V restricted to $X\backslash\bar{L}$. By Lemma 5 there exists a continuation V' of the data to X. Since V, V' are equivalent in $X\backslash\bar{B}$, they are also equivalent in $X\backslash\bar{A}$ (by Lemma 4).

Rothstein [5, Satz I*] stated a similar theorem for $n=3$, replacing "n-convex" by "analytic polyhedron" and omitting the condition (P), but in his proof 2 there occurs a serious mistake. The same remark applied to his treatment of the first Cousin problem in [6].

From Theorem 1 we get:

THEOREM 2. *Let X be a Cousin* II *domain in $\mathbf{C}^n$ and let A be an n-convex subset of X having the property* (P). *Then $X\backslash\bar{A}$ is a Cousin* II *domain.*

LEMMA 7. *Let X be an open set on a real n-dimensional differential manifold satisfying $H^q(X,\mathbf{Z})=0$ for $q=1,\cdots,m$ $(m<n)$, and let A be a contractible relatively compact subset of X with continuously differentiable boundary. Then $H^q(X\backslash\bar{A},\mathbf{Z})=0$ for $q=1,\cdots,m$.*

PROOF. Write $H^q(N)$ for $H^q(N, \mathbf{Z})$. Since ∂A is differentiable, $X\backslash\bar{A}$ can be deformed continuously to an open set B which contains ∂A. We have $H^r(X\backslash\bar{A})=H^r(B)$ for $r\geqq 0$. Since A is contractible, $H^r(A)=0$ if $r>0$. Next, $A\cap B$ can be deformed continuously to ∂A and, therefore, $H^r(A\cap B)=H^r(\partial A)$. By Lefschetz Duality Theorem [2], $H^r(A, \partial A)=0$ if $0\leqq r<n$, and from the exact sequence $H^r(A)\to H^r(\partial A)\to H^r(A, \partial A)$ we then infer that $H^r(\partial A)=0$; hence $H^r(A\cap B)=0$ if $0<r<n$. Noting that $A\cap B\neq\emptyset$, we can write down the Mayer-Vietoris exact sequence $H^r(X)\to H^r(A)\oplus H^r(B)\to H^r(A\cap B)$ and obtain $H^r(B)=0$ if $1\leqq r\leqq m$.

THEOREM 3. *Let X be a Cousin* I *domain in $\mathbf{C}^n$ and let A be an n-convex subset of X having the property* (P). *If $H^q(X, \mathbf{Z})=0$ for $q=1, 2$ and if A is contractible, then $X\backslash\bar{A}$ is a Cousin* I *domain.*

Indeed, X is Cousin II and, by Theorem 2, also $X\backslash\bar{A}$ is Cousin II. Since, by Lemma 7, $H^1(X\backslash\bar{A}, \mathbf{Z})=0$, $X\backslash\bar{A}$ is Cousin I.

COROLLARY. *If X is a domain of holomorphy in $\mathbf{C}^n$, if $H^q(X, \mathbf{Z})=0$ for $q=1, 2$, and if A is as in Theorem 3, then $X\backslash\bar{A}$ is both Cousin* I *and Cousin* II.

Theorems 1–3 extend to the case where instead of one hole A there is a finite number of holes. The results also extend to sets X on complex manifolds, provided B (in (P)) lies in one coordinate patch.

Added in proof. (I) Define "real $2q$-convex homotopic" analogously to "q-convex homotopic" by requiring the manifolds to be strictly convex in at least $n-q+1$ complex directions of the tangent hyperplanes. *Theorems* 1–3 *remain true if the condition (P) is relaxed by taking L to be real $2(n-1)$-convex homotopic to a point.* Indeed, modify the proof of Lemma 6 (L is strictly convex in z_2, z_3 directions) taking $K_j=L_j$ for $j=4, \cdots, n$ and, in the definition of A_1, $\epsilon<\mathrm{Re}(z_1)<d_0$. For fixed $\zeta\in W$, $\zeta\notin L$, take $S(t)$ to be 5-dimensional surfaces lying outside L, with $z_j=\zeta_j$ $(j=4, \cdots, n)$, $\zeta\in S(t)$ for some $t>1$, such that they are 1-convex. To construct $S(t)$ take E^j $(j=1, 2)$ convex 4-dimensional surfaces on $\mathrm{Re}(z_1)=-\beta_j$ $(\beta_1<\beta_2)$ and in G, and take 4-dimensional surface F, on $\mathrm{Re}(z_1)=-\beta_1$, lying outside L such that its intersection F_α with $\mathrm{Im}(z_1)=\alpha$ is convex for all small α. Let $E^1_\alpha=E^1\cap\{\mathrm{Im}(z_1)=\alpha\}$ and take $R^*=(\gamma, 0, \cdots, 0)(\gamma>0)$ outside L. $S(2)$ is a convex cap with top R^*, base E^2, and passing through E^1. Deform E^1 into F (by deforming E^1_α into F_α) and, correspondingly, deform the "meridians" issuing from R^* to E^2. $S(t)$ is the deformation of $S(2)$ at stage t $(1\leqq t<2)$.

(II) In [Math. Ann. **120** (1955), 96–138] Rothstein gave a proof of Theorem 1 with "n-convex" replaced by "$(n-1)$-convex" and with "(P)" replaced by "A is a star domain." His proof applies also to continuation of analytic sets, but our proof is much simpler.

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