HERGLOTZ TRANSFORMATION AND H^p THEORY

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Considerable work has been done in recent years to generalize a certain part of the theory of holomorphic functions on the unit disc, related to the Hardy classes H^p, as well as to determine a proper setting for the resulting abstract H^p theory. The reader is referred to the excellent accounts in Hoffman [5] and Hoffman-Rossi [6]; see also Lumer [7] and Srinivasan-Wang [9].

In this note we intend to describe how an approach to the above theory at its most general level, based on what we call the Herglotz transformation,[2] leads to: (i) advantages in deriving the theory; (ii) stronger and more explicit forms of known results; (iii) new results, which we believe shed light on the nature of the subject and yield applicable information. The Herglotz transformation corresponds in the classical case to the integral transformation defined by what is sometimes called the Herglotz kernel. In §1 we introduce the Herglotz transformation under most general conditions, and show that if A is any subalgebra of $L^\infty(m)$ containing 1, m a positive measure multiplicative on A, then a certain form of logmodularity will hold even if m is not unique as a representing measure on A. From this one can, in particular, settle a question implicitly left open by Hoffman-Rossi [6]: whether to simply assume density of Re A in $L^p_R(m)$ for all p finite will yield the usual H^p theory and imply the uniqueness of m in the sense of [6] (it is shown in [6] that density of Re A in L^1_R or L^2_R will not suffice). The answer is affirmative. In §2 we extend the transformation under uniqueness assumption on m, and use this in §3 to derive the full H^p theory. Once the properties of the transformation are known, many results will follow very directly without need of treating separately H^p spaces for different values of p. The crucial fact that Re A is dense in L^p_R for all p finite (under uniqueness assumption on m) is proved here without appeal to some nontrivial tool foreign to H^p theory (such as Lemma 6.6, [5]). As an example of "more precise forms of results," we prove a Szegö theorem giving not

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[2] Related ideas were used recently, in connection with special questions, by Devinatz [3], [2], dealing with Dirichlet algebras, and by Lumer [8], in a setting without uniform approximation assumptions.

only the minimum distance, but explicitly the projection, as well. One derives a computable expression for the prediction function in the linear prediction problem of [4]. Finally we deal with the following important question: to what extent is it true that if $f \in H^1$, and $F(z)$ is holomorphic on the "spectrum" (to be defined) of f, then $F(f)$ is in H^p provided it is in L^p. We define a spectrum $\sigma(f)$, for $f \in H^1$, and show that if $f, g \in H^1$, $R(z)$ is rational with poles outside $\sigma(f)$, then $gR(f)$ is in H^p provided it is in L^p, $p \geqq 1$. f outer implies $0 \notin \sigma(f)$, and we have some particular cases which appear in the literature, [5], [2], [3], [8].

1. **Herglotz transformation and logmodularity.** Let A denote a closed subalgebra of $L^\infty(m)$ containing 1, where m is a multiplicative measure on A [5]. For $S \subset L^\infty(m)$, S_R denotes the set of real-parts of elements in S.

DEFINITION 1. For $p > 0$, $H^p = H^p(m)$ is the closure of A in $L^p(m)$; H^∞ is $H^2 \cap L^\infty$. $\mathfrak{L}_R^\infty$ is defined as the L^ω-closure of A_R in L^∞.[3]

m is multiplicative on A, so that f real-valued in A and $\int f\, dm = 0$, imply $f = 0$ a.e. Hence there is no ambiguity in the following

DEFINITION 2. For $u \in A_R$, $T_0 u$ denotes the element of A such that a.e. $u = \mathrm{Re}(T_0 u)$, and $\int \mathrm{Im}(T_0 u)\, dm = 0$. T_0 shall be called the elementary Herglotz transformation.

T_0 is linear on A_R. Using a method introduced by Bochner [1], [2], one obtains

PROPOSITION 3. *Given $p \geqq 2$, there is a constant C_p such that $\forall u \in A_R$,*

$$\|T_0 u\|_p \leqq C_p \|u\|_p. \tag{1}$$

LEMMA 4. *The constants C_p in* (1) *can be chosen so that*

$$C_p = O(p) \quad \text{as } p \to \infty. \tag{2}$$

Because of (1), T_0 has an extension T to $\mathfrak{L}_R^\infty$, defined independently of p. The range of T is in $H^\omega = \bigcap_{p<\infty} H^p$, and we have $\mathrm{Re}(Tu) = u$, $\|Tu\|_p \leqq C_p \|u\|_p$, for $u \in \mathfrak{L}_R^\infty$ and each p finite $\geqq 2$.

THEOREM 5. *If $u \in \mathfrak{L}_R^\infty$, then* $\exp(Tu)$ *is an invertible element of H^∞, and*

$$u = \log|\exp(Tu)|. \tag{3}$$

SKETCH OF PROOF. H^∞ and H^ω are algebras, and for $u \in \mathfrak{L}_R^\infty$,

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[3] In accordance with a terminology introduced by Arens, L^ω denotes the topological algebra obtained by providing the intersection of all L^p for p finite (which is an algebra under point-wise multiplication), with the locally convex topology defined by the family of all L^p norms for p finite.

$$\|Tu\|_p \leqq C_p\|u\|_p \leqq Kp\|u\|_\infty \tag{4}$$

by virtue of Lemma 4, for p finite $\geqq 2$. In the power series in t (real)

$$\sum_{n=0}^{\infty} (\|(Tu)^n\|_2/n!)t^n \tag{5}$$

we have $\|(Tu)^n\|_2 = \|Tu\|_{2n}^n$, and $(n!)^{1/n} \sim n/e$ asymptotically. It will follow from (4) that (5) has a nonzero radius of convergence. It can then be derived that $\exp(t(Tu)) \in H^2$ for all $|t| \leqq$ some positive r. Since H^∞ is an algebra we see at once that $\exp(Tu) \in H^\infty$.

COROLLARY 6. *If A_R is dense in L_R^p for all p finite, then H^∞ is (in the sense of Hoffman [5]) a logmodular Banach algebra on the maximal ideal space of L^∞.*

Thus, the L^ω-density of A_R in L_R^∞ implies the uniqueness condition discussed by Hoffman-Rossi in [6], and all of the H^p results.

2. **Herglotz transformation and uniqueness.** Now let C be a linear subspace of $L^\infty(m)$ containing A, and denote by $[m]$ the functional defined on A_R as $[m](u) = \int u\, dm$, $\forall u \in A_R$.

THEOREM 7. *Suppose $[m]$ has a unique positive extension to C_R. Then T_0 has a linear extension T_1 to C_R, such that $\forall u \in C_R$, $T_1 u \in \bigcap_{p<1} H^p$, $u = \mathrm{Re}(T_1 u)$, and for each $p<1$, there is a constant C_p such that $\|T_1 u\|_p \leqq C_p\|u\|_1$ provided we have also $u \geqq 0$. Finally $\forall u \in C_R$, $\exp(T_1 u)$ is in H^∞.*

If in addition C_R is a sublattice of L_R^∞ (with the usual "sup" operation), then for each $p<1$, there exists K_p such that for every $u \in C_R$,

$$\|T_1 u\|_p \leqq K_p\|u\|_1. \tag{6}$$

3. **The H^p theory.** In order to develop the usual H^p theory we must now assume that the C introduced above is "large enough." In the usual situations C is either $C(X)$, the complex continuous functions on a compact Hausdorff space X, or just $L^\infty(m)$. From here on we shall thus assume:

(a) $[m]$ has a unique positive extension to an L^1-dense linear sublattice C_R of L_R^∞.[4]

We indicate briefly how the usual H^p theory is derived in this approach. This applies at once to the case $C = C(X)$, m a unique representing measure, and also readily to the case in which we assume weak* (or only L^ω) density of Re A in L_R^∞. To show that the theory

[4] It is well known that the L^1 density of C_R alone, is not sufficient to insure the validity of the usual H^p results.

developed in [6] by Hoffman and Rossi reduces to the above requires the use of essentially the first three lemmas in their paper. Under assumption (a) it follows that T_0 has a linear extension T to L^1_R having all the properties stated in Theorem 7 for T_1. T will be a basic tool for the H^p theory. As usual we set $A_m = \{f \in A : \int f\, dm = 0\}$. We use temporarily the notation $\tilde{H}^p = \{f \in L^p : \int fg\, dm = 0 \forall g \in A_m\}$.

THEOREM 8. *If $u \in L^1_R$, and $g \in \tilde{H}^p$, then $g \exp(Tu) \in \tilde{H}^q$ provided it is in L^q, p and $q \geqq 1$. If g is in H^p, then, provided it is in L^q, $g \exp(Tu) \in H^q$ and $\int g \exp(Tu) dm = (\int g\, dm) \exp(\int u\, dm)$. If g is in addition outer, then $g \exp(Tu)$ is outer.*

It can now be shown from Theorem 8 and usual arguments, but before any machinery is developed, that $\tilde{H}^2 = H^2$ and $\tilde{H}^1 = H^1$. From this we come to the crucial fact that Re A is dense in L^p_R, p finite, as follows

THEOREM 9. *If $f \in H^1$ and is real-valued then f is constant a.e.*

PROOF. Suppose $u_n + iv_n = f_n \in A$, and $\|f - f_n\|_1 \to 0$. Since f is real, $\|v_n\|_1 \to 0$, and $\|f - u_n\|_1 \to 0$. We may assume $\int f\, dm = 0$, and hence may also assume $\int u_n\, dm = 0$. Hence $Tv_n = if_n$ tends to 0 in L^p for $p < 1$, by Theorem 7, and from $\|f - u_n\|_1 \to 0$, follows then that $f = 0$ a.e.

COROLLARY 10. *A_R is weak* dense in L^∞_R, hence norm-dense in L^p_R for $p < \infty$.*

PROOF. If $f \in L^1_R$ and $\int fg\, dm = 0$, $\forall g \in A$, then $f \in \tilde{H}^1_m = H^1_m$, and being real, is 0 by Theorem 9.

From this, and Proposition 3 completed by duality for $1 < p < 2$,

COROLLARY 11. *If $u \in L^p_R$, p finite > 1, then $Tu \in H^p$.*

COROLLARY 12. *If $f \in H^1$, and $f \in L^p$, $1 < p \leqq \infty$, then $f \in H^p$.*

PROOF. Re $f \in L^p$, hence $g = T(\text{Re } f) \in H^1$. Thus $i(f - g)$ is real-valued and in H^1, hence constant by Theorem 9. It follows that $f \in H^p$.

Many results follow now quite rapidly, for instance,

THEOREM 13. *If $f \in H^1$, $\log |f| \in L^1$, the outer part of f is but for a constant factor of modulus one, given by $F = \exp(T(\log |f|))$, and $f = FJ$, where J is inner.*

PROOF. From Theorem 8, F is outer; and since $J = f \exp(-T(\log |f|)) \in L^\infty$, we have $J \in H^\infty$ from Theorem 8. Also $|J| = 1$, a.e., so that J is inner.

THEOREM 14. *Let* $f \in H^1$, $\log |f| \in L^1$, p *and* $q \geqq 1$ *be such that* $1/p + 1/q = 1$. *Then* $f = hg$, *where* $h \in H^p$ *and* $g \in H^q$.

PROOF. By Theorem 13, we may assume f outer and $\int f\, dm$ real, > 0. Then $f = \exp(T(\log|f|))$. Set $h = \exp(T(\log|f|/p))$, $g = \exp(T(\log|f|/q))$. Then $h \in L^p$, $g \in L^q$, and by Theorem 8 we have $h \in H^p$, $g \in H^q$; and surely $f = hg$.

4. **The Szegö theorem and linear prediction.** The following is a stronger version of the usual Szegö theorem (see also [9]).

THEOREM 15. *Let* $h \geqq 0$ *be in* L^1, p *be finite and* $\geqq 1$. *Then the minimum distance of the constant function* 1 *to the closure* S *of* A_m *in* $L^p(hdm)$, *is achieved at*

$$1 - \exp\left(1/p\left(\int \log h\, dm - T(\log h)\right)\right) \tag{7}$$

if $\log h \in L^1$, *and at* 1 *itself otherwise. The minimum distance is given by* $(\exp(\int \log h\, dm))^{1/p}$.

PROOF. Set f for the exponential which appears in (7), where we assume that $\log h \in L^1$. Clearly f is in $L^p(hdm)$. If $1-f$ is not in S, there is a g in $L^q(hdm)$, $1/p+1/q=1$, such that $\int(1-f)gh\, dm \neq 0$, while g is "orthogonal" to A_m. The latter implies, since $gh \in L^1$, that $gh \in H^1$. Hence we can apply Theorem 8, after checking that $ghf \in L^q(m)$, to see that $ghf \in H^q$. Similarly, $f^{-1} \in L^p(m)$ yields $f^{-1} \in H^p$. Also Theorem 8 shows that $\int f^{-1}\, dm = 1$. This implies $\int gh\, dm = \int f^{-1}\, dm \int ghf\, dm = \int ghf\, dm$, a contradiction. This proves that $(1-f) \in S$, so that the minimum distance is $\leqq \|f\|_p = (\exp(\int \log h\, dm))^{1/p}$. The reverse inequality, and the case $\log h \notin L^1$, are not difficult to handle from here on.

We can apply this to the solution of the linear prediction problem, in the setting of Helson-Lowdenslager [4], considering for instance, as they do, a doubly stationary sequence. To the latter corresponds a correlation function which is positive definite as a function on the discrete group Z^2, and thus the Fourier transform of a positive measure μ on the 2-torus. We assume that the subset of Z^2 for which the prediction problem is to be solved is a half-plane in the sense of Helson-Lowdenslager [4]. Via a certain isometric map (see [4]) the solution of the prediction problem corresponds to a function Φ in $L^2(\mu)$. If m denotes the normalized Haar measure on the 2-torus, and h the Radon-Nikodym derivative of μ with respect to m, then the prediction error, as given in [4] is $(\exp(\int \log h\, dm))^{1/2}$. Now Theorem 15 enables us to write down Φ as

$$\Phi = 1 - \exp\left(1/2\left(\int \log h \, dm - T(\log h)\right)\right) \tag{8}$$

from which approximations to Φ can be computed.

5. **Holomorphic functions of H^p elements.** We need a notion of spectrum.

DEFINITION 16. If $f \in H^1$, the inner spectrum (we shall usually say spectrum) $\sigma(f)$ of f, is defined by

$$\sigma(f) = \{\lambda \text{ complex: } \lambda - f \text{ is not outer}\}. \tag{9}$$

THEOREM 17. *If $f \in H^1$, $g \in H^1$, and $R(z)$ is a rational function of the complex variable z with poles outside $\sigma(f)$, then $gR(f) \in H^p$, $p \geqq 1$, provided it is in L^p.*

COROLLARY 18. *If $f \in H^1$ is outer, $g \in H^1$, and $gf^{-1} \in L^p$, $1 \leqq p \leqq \infty$, then $gf^{-1} \in H^p$. In particular, if $|g| \leqq |f|$, then $gf^{-1} \in H^\infty$; if $|f|^{-1} \in L^1$, it is in H^1.*

THEOREM 19. *If $f \in H^1$, then $\sigma(f)$ is contained in the convex hull of the set of values of f.*

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