RESEARCH ANNOUNCEMENTS

The purpose of this department is to provide early announcement of significant new results, with some indications of proof. Although ordinarily a research announcement should be a brief summary of a paper to be published in full elsewhere, papers giving complete proofs of results of exceptional interest are also solicited. Manuscripts more than eight typewritten double spaced pages long will not be considered as acceptable.

A THEOREM ON MONOTONICITY AND A SOLUTION TO A PROBLEM OF ZAHORSKI

BY A. M. BRUCKNER[1]

Communicated by R. P. Boas, June 7, 1965

In the body of his study of derivatives [5], Zahorski posed two problems. The second of these problems is the following:

ZAHORSKI'S PROBLEM II. Let f be a finite real valued function defined on an interval I_0 and satisfying the following conditions:

(1) f is a Darboux function,

(2) f is in the first class of Baire,

(3) f possesses an approximate derivative f'_{ap}, finite or infinite, except perhaps on a denumerable set,

(4) $f'_{ap} \geqq 0$ almost everywhere.

Is it necessarily true that f is continuous and nondecreasing on I_0?

Zahorski notes that Tolstoff [3] proved that if conditions (1) and (2) are replaced by the more stringent condition that f be approximately continuous, while conditions (3) and (4) remain the same, then f must be continuous and nondecreasing. Furthermore, a slight modification of an example found in [4] shows that condition (2) in the statement of Zahorski's problem cannot be weakened to the requirement that f be in the second class of Baire.

After stating Problem II, Zahorski proceeded to prove that if the approximate derivative in conditions (3) and (4) is replaced by the ordinary derivative, then f must be continuous and nondecreasing. (In this case condition (2) is an automatic consequence of condition (3).)

It is clear that an affirmative answer to Zahorski's Problem II represents a generalization of both Tolstoff's Theorem and Zahorski's

[1] This work was accomplished while the author was under the support of NSF grant GP 1592.

Theorem. The major purpose of the present note is to obtain a theorem which implies such an affirmative answer (Theorem 5, below).

To obtain this result we need four preliminary theorems. Most of the concepts and notations appearing in the statements of these theorems can be found in Saks [2], particularly Chapters VII and IX.

THEOREM 1. *Let f be a Darboux function in Baire class* 1 *and satisfying Banach's condition T_2 on an interval I_0. Then, for almost all y in $f(I_0)$, the level set $E_y \equiv \{x: f(x) = y\}$ either consists of a single point, or contains a pair of isolated neighbors. (The points x_1 and x_2 in E_y are said to be isolated neighbors of E_y provided x_1 and x_2 are isolated in E_y and $(x_1, x_2) \cap E_y$ is empty.)*

THEOREM 2. *Let f be a Darboux function in Baire class* 1 *and satisfying Banach's condition T_2 on $I_0 = [a, b]$. Let $P = \{x: 0 \leqq f'(x) \leqq \infty\}$. If $f(a) < f(b)$ then P is nondenumerable, $f(P)$ is measurable, and $mf(P) \geqq f(b) - f(a)$. If $f(a) > f(b)$, then similar statements are valid for the set $N = \{x: -\infty \leqq f'(x) \leqq 0\}$.*

Theorems 1 and 2 are not valid if we weaken the requirement that f be a Baire 1 function to the requirement that f be a Baire 2 function. In fact, with the aid of the Cantor set, it is easy to construct a function f on $[0, 1]$ with the following properties:

(a) $f(0) = 0, f(1) = 1$.

(b) f is a Darboux function in Baire class 2 and satisfying Banach's condition T_2.

(c) For every y, $0 < y < 1$, the set E_y is an isolated set which is order isomorphic to the rationals (i.e., between any two members of E_y there is another).

(d) The set P of Theorem 2 is empty.

THEOREM 3. *Let f be a Darboux function in Baire class* 1 *and satisfying Banach's condition T_2 on I_0. Then for every pair of real numbers $\alpha < \beta$, the set $E_{\alpha\beta} \equiv \{x: \alpha < f(x) < \beta\}$ is either empty or contains an interval.*

THEOREM 4. *Let f be a Darboux function in Baire class* 1 *on I_0. If f is of generalized bounded variation (VBG) on I_0, then there exists a sequence of intervals $\{I_n\}$ whose union is dense in I_0 and on each of which f is continuous and of bounded variation.*

We turn now to our main result, a result which implies an affirmative answer to Zahorski's Problem II.

THEOREM 5. *Let $\mathcal{P}$ be any function-theoretic property sufficiently* *strong to imply conditions* (i) *and* (ii) *below:*

(i) *If f is continuous and of bounded variation on an interval I_0 and possesses property $\mathcal{P}$ on I_0, then f is nondecreasing on I_0.*

(ii) *If f is a Darboux function in Baire class* 1 *and possesses property $\mathcal{P}$ on I_0, then f is VBG on I_0.*

Then any Darboux function in Baire class 1 *which possesses property $\mathcal{P}$ on I_0 is continuous and nondecreasing on that interval.*

To see that Zahorski's Problem II has an affirmative answer, we let $\mathcal{P}$ be the property of having, with the possible exception of a denumerable set of points, an approximate derivative (finite or infinite) which is nonnegative almost everywhere. That Condition (i) is then satisfied is a consequence of Tolstoff's Theorem, and that Condition (ii) is met is a consequence of Theorem 10.8 of Chapter VII of Saks [2].

In general, Theorem 5 can be used in the following way. If we wish to show that a certain condition (probably involving some sort of generalized derivative being nonnegative on a "large" set) is sufficiently strong to force a Darboux Baire 1 function to be nondecreasing, we need only check that it is sufficiently strong to force a continuous function of bounded variation to be nondecreasing and that it is sufficiently strong to force a Darboux Baire 1 function to be *VBG*.

We now proceed to outline the main steps in the proof of Theorem 5. We omit computational details.

Let $\mathcal{P}$ be a property satisfying the hypotheses of Theorem 5 and let f be a Darboux Baire 1 function possessing property $\mathcal{P}$ on I_0.

(1) From Theorem 4 and the hypothesis of Theorem 5 we infer the existence of a sequence of intervals $\{I_n\}$ whose union is dense in I_0 and on each of which f is continuous and nondecreasing. We take the intervals to be maximal open intervals of monotonicity. We wish to show the sequence $\{I_n\}$ reduces to a single interval, $\mathring{I}_0$, the interior of I_0.

(2) Let $P=\mathring{I}_0-\bigcup_n I$. If there is more than one interval in the sequence $\{I_n\}$, then P is a nonempty perfect set. The function f is *VBG* on P, from which we infer the existence of a set H and an interval $J\subset I_0$ such that H is dense in $P\cap J$ and f is *VB* on H.

(3) The function f is, in fact, *VB* on $P\cap J$. This part of the proof is rather lengthy and involves showing that if $x_0\in P\cap J$, if J_1 is a subinterval of J containing x_0 and if $\alpha<f(x_0)<\beta$, then there is a point $h\in H\cap J_1$ such that $\alpha<f(h)<\beta$. We establish this fact in the following manner. First we find points $z_1<z_2$ in $J_1\cap E_{\alpha\beta}$ such that

$f(z_1) > f(z_2)$. We then use Theorem 2 to show that the set $Q \equiv \{x: -\infty \leq f'(x) < 0\} \cap [z_1, z_2] \cap E_{\alpha\beta}$ is a nondenumerable subset of P. The set Q is a Borel set on which f is VBG_*. It follows that there is a nonempty perfect subset S of Q and an $\epsilon > 0$ such that (i) f is VB_* on S and (ii) for $s \in S$, $\alpha + \epsilon < f(s) < \beta - \epsilon$. Since S is a perfect subset of P, it follows from (i) and (ii) that there is an $h \in H \cap J_1$ such that $\alpha < f(h) < \beta$. It now follows readily that f is VB on $P \cap J$.

(4) The function f is of bounded variation on J. This result is a simple consequence of (3) and the monotonicity of f on each interval complementary to P.

(5) We infer from (4) that f is continuous on J (any Darboux function of bounded variation on an interval J is continuous on J). But the hypotheses of Theorem 5 now imply f is nondecreasing on J, which contradicts the maximality of the intervals of the sequence $\{I_n\}$. This completes the proof of Theorem 5.

Theorem 5, or, more specifically, the affirmative answer to Zahorski's Problem II, can be used to obtain various other results. For example, if a Darboux Baire 1 function has everywhere on I_0 an approximate derivative f'_{ap} such that f'_{ap} is also in Baire class 1, then f'_{ap} is itself a Darboux function. (The function f'_{ap} is always in Baire class 2.) This generalizes a result of Kulbacka [1]. Theorem 5 can also be used to characterize the stationary and determining sets of certain classes of functions. We do not go into the details here—the results will appear in a subsequent article.

References

1. M. Kulbacka, *Sur certaines propriétés des derivées approximatives*, Bull. Acad. Polon. Sci. Sér. Sci Math. Astronom. Phys. 12 (1964), 17–20.

2. S. Saks, *Theory of the integral*, Vol. 7, Monografie Matematyczne, Warszawa-Lwow, 1937.

3. G. Tolstoff, *Sur quelques propriétés des fonctions approximativement continues*, Rec. Math. (Mat. Sb.) N.S. 5 (1939), 637–645.

4. Z. Zahorski, *Sur la classe de Baire des dérivées approximatives d'une fonction quelconque*, Ann. Soc. Polon. Math. 21 (1948), 306–323.

5. ———, *Sur la première dérivée*, Trans. Amer. Math. Soc. 69 (1950), 1–54.

University of California, Santa Barbara