SOME SPACES WHOSE PRODUCT WITH E^1 IS E^4

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Communicated by O. G. Harrold, March 22, 1965

1. **Introduction.** If A is a collection of subsets of E^3, then $A^* = \bigcup\{a \mid a \in A\}$. A sequence A_i, $i = 1, 2, 3, \cdots$, of locally finite disjoint collections of subsets of E^3 is trivial if $A_{i+1}^* \subset \operatorname{Int}(A_i^*)$, each element of A_i is a cube with handles semi-linearly imbedded in E^3, and the inclusion map $j: a' \to a$, where $a' \subset a \in A_i$ and $a' \in A_{i+1}$, is null homotopic.

If A_i, $i = 1, 2, \cdots$, is a trivial sequence let G be the set of points of $E^3 - \bigcap A_i^*$ and components of $\bigcap A_i^*$. Let X be the corresponding decomposition space. The main result, Theorem 2, may now be stated.

THEOREM 2. *If each element of A_i, $i = 1, 2, \cdots$, is a solid torus, then $X \times E = E^4$.*

This theorem is parallel to results in [1], [3], [4] and others. The proof is similar to that given in [4].

2. **Some useful maps.** Let $D = \{z \mid z \in E^2 \text{ and } |z| \leq 1\}$, $S = \{z \mid z \in E^2 \text{ and } |z| = 1\}$, $D_1 = \{z \mid z \in E^2 \text{ and } |z| \leq 1/2\}$, $T = D \times S$ and $B = D_1 \times S \subset T$. Let $p: D_1 \times E \to B$ be the universal covering of B where p is given by $p(x, t) = (x, e^{it})$ for $x \in D_1$, $t \in E$. Let $h: D_1 \times E \to T \times E$ by $h(x, t) = (x, e^{it}, t)$ and $q: T \times E \to T$ by $q(x, s, t) = (x, s)$ where $x \in D$, $s \in S$ and $t \in E$. Hence $qh(x, t) = p(x, t)$.

Let B' be a finite subcomplex of Int (B) such that the inclusion map $j: B' \to \operatorname{Int}(B)$ is null homotopic. Using the homotopy lifting theorem, there exists $j^*: B' \to D_1 \times E$ such that:

$$\begin{tikzcd} & D_1 \times E \arrow[r, "h"] \arrow[d, "p"] & T \times E \arrow[d, "q"] \\ B' \arrow[ur, "j^*"] \arrow[r, "j"'] & B \arrow[r, phantom, "\subset"] & T \end{tikzcd}$$

is commutative and both j^* and h are homeomorphisms.

If $u \in B'$, $hj^*(u) = (u, \psi(u))$ where $\psi: B' \to E$. If $(x, s) \in B'$ where $x \in D_1$, $s \in S$, then $j^*(x, s) = (x, w(x, s))$ where $w: B' \to E$. By commutativity,

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[1] Supported in part by NSF GP 4006.

$$(*) \qquad e^{w(x,s)i} = s \quad \text{and} \quad \psi(x, s) = w(x, s).$$

Since B' is compact, $\psi(B') \subset [-\alpha, \alpha]$, for some $\alpha \in E$. We extend ψ to a map Ψ: $T \to [-\alpha, \alpha]$ by:

$$\Psi(u) = \psi(u) \quad \text{for } u \in B',$$
$$\Psi(u) = 0 \quad \text{for } u \in T - B,$$

$\Psi(u)$ is determined by Tietze's extension theorem otherwise.

Define the homeomorphism λ: $T \times E \to T \times E$ by $\lambda(u, t) = (u, \Psi(u) + t)$. We call λ the lifting map; it will be used later.

Let T_ϕ: $T \to T$ be a homeomorphism of T onto T defined for each real number ϕ by the following:

$$T_\phi(x, s) = \begin{cases} (x, s) & (x, s) \in \operatorname{Bd}(T), \\ (x, se^{i(2\phi|x| - 2\phi)}) & (x, s) \in T - (B \cup \operatorname{Bd}(T)), \\ (x, se^{-i\phi}) & (x, s) \in B \end{cases}$$

where $x \in D$ and $s \in S$. Let τ: $T \times E \to T \times E$ by $\tau(u, t) = (T_t(u), t)$. We call τ the twisting map.

Consider $q\tau\lambda$: $T \times E \to T$. For $x \in D$, $s \in S$ and $t \in E$, $q\tau\lambda(x, s, t) = T_{\Psi(x,s)+t}(x, s)$. If $(x, s) \in B'$ then

$$q\tau\lambda(x, s, t) = (x, se^{-(\Psi(x,s)+t)i}) = (x, e^{-ti})$$

by (*).

LEMMA 1. *The homeomorphism $f = \tau\lambda$: $T \times E \to T \times E$ has the following properties*:

(1) $f = \text{id.}$ *on* $\operatorname{Bd}(T \times E)$,

(2) $\operatorname{diam}(f(B' \times w)) \leqq \operatorname{diam}(D_1 \times [-\alpha, \alpha])$,

(3) $f(T \times w) \subset T \times [w - \alpha, w + \alpha]$ *and*

(4) *f is uniformly continuous on $B \times E$.*

PROOF. We shall only prove (4), the other three following easily from the construction. Let $\epsilon > 0$. For the uniformly continuous map $\bar{f} = f | B \times [0, 4\pi]$, let $0 < \delta < 2\pi$ be as in the definition of uniform continuity. If $u, v \in B$, $t \leqq s \in E$ and $d((u, t), (v, s)) < \delta < 2\pi$, let k be an integer chosen so that $0 \leqq t - 2k\pi < 2\pi$ and $0 \leqq s - 2k\pi < 4\pi$. By uniform continuity, $d(\bar{f}(u, t - 2k\pi), \bar{f}(v, s - 2k\pi)) < \epsilon$. But $d(f(u, t), f(v, s)) = d(\bar{f}(u, t - 2k\pi), \bar{f}(v, s - 2k\pi))$; hence f is uniformly continuous on $B \times E$.

3. **Shrinking sets in a solid torus.** Assume now that each element of A_i, $i = 1, 2, \cdots$, is a solid torus.

LEMMA 2. *Let $A \in A_i$, $A_0 = A^*_{i+1} \cap A$ and $\epsilon > 0$. There exists a homeo-*

morphism H of E^4 onto itself such that:

(1) $H=\text{id.}$ *on the complement of* $A\times E$,

(2) $\operatorname{diam}(H(A_0\times w))<\epsilon$,

(3) $H(A\times w)\subset A\times[w, w+\epsilon]$ *and*

(4) *H is uniformly continuous on* $A_0\times E$.

PROOF. Let g' be a homeomorphism of A onto T; we assume that g' is chosen so that $g'(A_0)\subset\operatorname{Int}(B)$. Define $g: A\times E\to T\times E$ by $g(x, t) =(g'(x), t)$. For g^{-1} and $\epsilon>0$ let δ be as in the definition of univorm continuity. Now make the following modifications of the function f of Lemma 1. The disk D_1 could have been chosen to be of arbitrarily small diameter; similarly the function Ψ could have been chosen so that α would be arbitrarily small. Then we may assume that $\operatorname{diam}(D_1\times[-\alpha, \alpha])<\delta$, and furthermore that $\alpha<\epsilon/2$. Then define a homeomorphism $f': T\times E\to T\times E$ by $f'(x, t)=(y, s+\epsilon/2)$ where $f(x, t)=(y, s)$. Defining $H=\text{id.}$ outside $A\times E$ and $h^{-1}f'h$ elsewhere, then H is as desired.

One now proceeds as in [2] or [4] to derive Theorem 1 and Theorem 2.

THEOREM 1. *There exists a uniformly convergent sequence of homeomorphisms of $E^3\times E$ onto $E^3\times E$ whose limit f satisfies*:

(1) *f is 1-1 outside* $\bigcap A_i^*\times E$,

(2) *$f(g\times w)$ is a point for each $g\in G$ and*

(3) *If $g\times w\neq g'\times w'$, then $f(g\times w)\neq f(g'\times w')$ for $g\in G$.*

THEOREM 2. *If each element of A_i, $i=1, 2, \cdots$, is a solid torus, then* $X\times E=E^4$.

Theorem 2 of [4] is a corollary of our Theorem 2; the process of lifting and twisting is essentially the method used there. One can see that although Theorem 2 is quite general, it is a special case of the following conjecture.

CONJECTURE. If each element of A_i, $i=1, 2, \cdots$, is a cube with handles, then $X\times E=E^4$.

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