THE EXISTENCE OF COMPLETE CYCLES IN REPEATED LINE-GRAPHS[1]

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With every nonempty ordinary graph G there is associated a graph $L(G)$, called the *line-graph* of G, whose points are in one-to-one correspondence with the lines of G and such that two points are adjacent in $L(G)$ if and only if the corresponding lines of G are adjacent. By $L^2(G)$, we shall mean $L(L(G))$; and, in general, $L^k(G)$ will denote $L(L^{k-1}(G))$ for $k \geqq 1$, where $L^1(G)$ and $L^0(G)$ stand for $L(G)$ and G, respectively. The graphs $L(G)$, $L^2(G)$, $L^3(G)$, etc. are referred to as the *repeated line-graphs* of G. A *complete cycle* (or *hamiltonian cycle*) in a (connected) graph G is a cycle containing all the points of G. The purpose of this note is to outline a proof of the following result, a complete proof of which will be published elsewhere.

THEOREM 1. *If G is a nontrivial connected graph of order p (having p points), and if G is not a path, then $L^n(G)$ contains a complete cycle for all $n \geqq p-3$. Furthermore, the number $p-3$ cannot, in general, be improved.*

A graph G having q lines, where $q \geqq 3$, is called *sequential* if the lines of G can be ordered as $x_0, x_1, x_2, \cdots, x_{q-1}, x_q = x_0$ so that x_i and x_{i+1}, $i = 0, 1, \cdots, q-1$, are adjacent. The next theorem follows immediately.

THEOREM 2. *A necessary and sufficient condition that the line-graph $L(G)$ of a graph G contain a complete cycle is that G be a sequential graph.*

If a graph G contains a complete cycle C, then the lines of C can be arranged in a cyclic fashion. By an appropriate "interspersing" of the lines not on C (if any) among the lines which are on C, we can produce an ordering of all the lines of G as needed to show that G is sequential. This fact coupled with Theorem 2 gives the next result.

THEOREM 3. *If a graph G contains a complete cycle, then $L(G)$ also contains a complete cycle.*

COROLLARY. *If a graph G contains a complete cycle, then $L^n(G)$ contains a complete cycle for all $n \geqq 1$.*

[1] This research is part of a doctoral thesis written under the direction of Professor E. A. Nordhaus of Michigan State University.

The following two lemmas can be quickly established.

LEMMA 1. *If a graph G has a cycle C with the property that every line of G is incident with at least one point of C, then $L(G)$ contains a complete cycle.*

LEMMA 2. *Let G be a graph consisting of a cycle C and its diagonals (a diagonal of C being a line which is not on C but which is incident with two points of C) and m paths $P_1, P_2, \cdots, P_m$, where* (i) *each path has precisely one endpoint in common with C and* (ii) *for $i \neq j$, P_i and P_j are disjoint except possibly having an endpoint in common if this point is also common to C. Then, if the maximum of the lengths of the P_i is M, $L^n(G)$ contains a complete cycle for all $n \geqq M$.*

The proof of Theorem 1 is by induction on p with the graphs having order 3, 4, or 5 treated individually. It is assumed then that for all connected graphs G' which are not paths and which have order s, where $s<p$ and $p \geqq 6$, $L^n(G')$ contains a complete cycle for each $n \geqq s-3$. The proof involves showing that if G is a graph which is not a path and which has order p, then $L^{p-4}(G)$ is a sequential graph so that $L^{p-3}(G)$ contains a complete cycle (by Theorem 2) and $L^n(G)$ contains a complete cycle for all $n \geqq p-3$ (by the corollary to Theorem 3).

If G is a cycle, the result follows directly, so without losing generality, we assume that G contains a point v having degree 3 or more. Let H denote the connected star subgraph whose lines are all those incident with v, and let Q denote the subgraph whose point set consists of all the points of G different from v and whose lines are all those which are in G but not in H. H and Q have deg v points in common but are line disjoint. We denote the components of Q by $G_1, G_2, \cdots, G_k$.

$L(H)$ is a complete subgraph of $L(G)$ and so has a cycle containing all the points of $L(H)$. If J_1 denotes $L(H)$ plus all those lines in $L(G)$ incident with one point of $L(H)$, then, by Lemma 1, $H_1 = L(J_1)$ has a cycle containing all the points of H_1. We let J_2 denote $L(H_1)$ plus any lines of $L^2(G)$ incident with a point of $L(H_1)$ and let $H_2 = L(J_2)$. Once again, by Lemma 1, H_2 has a cycle containing all the points of H_2. J_i and H_i, $i = 3, 4, \cdots$, are defined analogously, and each H_i has a cycle containing all the points of H_i.

Two cases are considered: (1) All the G_i are paths or isolated points, and (2) there is at least one G_i different from a path or an isolated point. In the first case, it follows, with the aid of Lemma 2, that $L^{p-4}(G)$ contains a complete cycle so that $L^{p-3}(G)$ contains such a cycle also.

In the second case, we assume that the first t components, $1 \leqq t \leqq k$, of $G_1, G_2, \cdots, G_k$ are not paths or isolated points. Clearly, each of the components $G_1, G_2, \cdots, G_t$ has at least 3 points. If $t<k$, the paths (or isolated points) $G_{t+1}, \cdots, G_k$ have orders at most $p-4$, and it is easily seen that for these components, $L^{p-4}(G_i)$ does not exist. $L^{p-4}(G)$ can thus be expressed as the pairwise line disjoint sum of the graphs J_{p-4}, $L^{p-4}(G_1)$, $L^{p-4}(G_2), \cdots, L^{p-4}(G_t)$, where each of the graphs $L^{p-4}(G_i)$, $i=1, 2, \cdots, t$, has a cycle containing all the points of $L^{p-4}(G_i)$ by the inductive hypothesis.

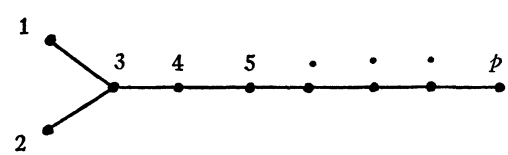
Since $p \geqq 6$, it can be shown that for each $i=1, 2, \cdots, t$, there is a point u_i in H_{p-4} adjacent to both endpoints of a line in $L^{p-4}(G_i)$. Using this result, we produce a suitable ordering of the lines of $L^{p-4}(G)$ thereby showing it to be a sequential graph.

Theorem 1 permits us to make the following definition.

DEFINITION. Let G be a nontrivial connected graph which is different from a path. The *hamiltonian index* of G, denoted by $h(G)$, is the smallest nonnegative integer n such that $L^n(G)$ contains a complete cycle.

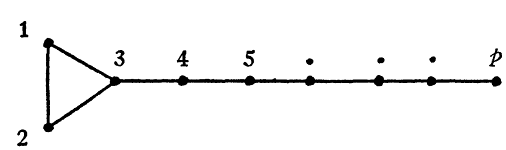
It now follows immediately that a graph contains a hamiltonian cycle if and only if its hamiltonian index is zero. Theorem 1 may now be restated in the following way. If G is a nontrivial connected graph of order p which is not a path, then $h(G)$ exists and $h(G) \leqq p-3$. To show that the bound given in Theorem 1 cannot be improved, we note that for every $p \geqq 3$, there are graphs whose hamiltonian indices are $p-3$. The graphs G_1 and G_2 shown in Figure 1 have hamiltonian indices equal to $p-3$.

G_1:



$p \geqq 4$

G_2:



$p \geqq 3$

FIGURE 1

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