SOME HOMOTOPY GROUPS OF STIEFEL MANIFOLDS[1]

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Paechter [7] made some computations of $\pi_{k+p}(V_{k+m,m})$ where $V_{k+m,m}$ is the Stiefel manifold of m frames in $k+m$ space. In this note we give a table (Table 1) extending his results in the case where m is large. Since $V_{k+m,m}\to V_{k+m+1,m+1}\to S^{k+m}$ is a fibering it is clear that $\pi_{k+p}(V_{k+m,m})$ depends only on k and p for $p\leqq m-2$. This is called the stable range and we feel that these stable groups are the most important ones. On the other hand for small values of m, one of us [4] has made extensive computations and the results are available.

James' periodicity [5, Theorem 3.1] is reflected in the table but the basic periodicity of period 8 is also present.

In [1] it is proved that if $n > 12$, then $\pi_j(SO(n)) = \pi_j(SO) + \pi_{j+1}(V_{2n,n})$ for $j<2n-1$. Hence it is easy to deduce the first fourteen nonstable groups of $SO(n)$ from this table.

Tables of homotopy groups are much more useful if generators are given. Instead of generators we settle for giving the order of the image of $i_*\colon \pi_{k+p}(S^k)\to\pi_{k+p}(V_{k+m,m})$ (Table 2). One can construct the generators from this information and this map has important connections with Whitehead products [2].

The groups have been computed by using modified Postnikov towers [6]. An outline of the computation for one case, 6 mod 32, is given. The case $k\equiv 6$ mod 32. This procedure is essentially the same as the Adams spectral sequence method.

Let $k=32n+6$ and we suppose m is large. Consider the fibering $V_{32n+6,7}\to V_{32n+m,m+1}\to V_{32n+m,m-6}$. We are only interested in groups in the homotopy stable range so that we can construct a new fibering

$$\Sigma^{-1}V_{32n+m,m-6}\to V_{32n+6,7}\to V_{32n+m,m+1}.$$

We will build the modified Postnikov tower to this fibering. By [3] the cohomology of $V_{32n+m,m+1}$ is given by

$$\begin{aligned} H^i(V_{32n+m,m+1};Z_2) &= 0, && 0<i<32n-1.\\ &= Z_2, && 32n-1\leqq i\leqq 32n+m-1.\end{aligned}$$

Let h_i generate $H^i(V_{32n+m,m+1};\,Z_2)$ when it is nonzero. Then Sq^jh_i

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TABLE 1

$\pi_{k+p}(V_{k+m,m})$ for m large and $k \equiv i$ mod 8 except as otherwise noted

p \ i	0	1	2	3	4	5	6	7
0	Z	Z_2	Z	Z_2	Z	Z_2	Z	Z_2
1	Z_2^2	Z_2	Z_4	0	Z_2^2	Z_2	Z_4	0
2	Z_2^2	Z_8	0	Z_2	Z_2^2	Z_8	0	Z_2
3	$Z_8^2+Z_3$	Z_2	Z_4+Z_3	Z_2^2	$Z_4+Z_{16}+Z_3$	Z_2	Z_4+Z_3	Z_2
4	Z_2	0	Z_2	Z_{16}	Z_2	0	0	Z_8
5	0	Z_2	Z_{16}	Z_2	0	0	Z_8	Z_2
6	Z_2^2	Z_2Z_{16}	Z_2^2	Z_2	0	Z_8	Z_4	0
7	0(16) $Z_{16}^2+Z_2+Z_{15}$ 8(16) $Z_{32}+Z_8+Z_2+Z_{15}$	Z_2^3	$Z_{16}+Z_2+Z_{15}$	Z_2	$Z_{16}+Z_4+Z_{15}$	Z_2^2	$Z_{16}+Z_{15}$	7(16) Z_2^2 15(16) Z_2
8	Z_2^5	Z_2^4	Z_4	Z_8	Z_2^4	Z_2^3	6(32) Z_4Z_2 22(32) Z_4^2 14(16) Z_4	7(16) Z_2Z_{32} 15(16) $Z_{16}Z_2$
9	Z_2^7	Z_8Z_2	Z_8	Z_2^2	Z_2^5	5(32) $Z_2^2Z_4$ 21(64) $Z_2+Z_4^2$ 53(64) $Z_2+Z_4+Z_8$ 13(16) Z_2Z_4	6(32) Z_2Z_{32} 22(32) Z_2Z_{64} 14(16) Z_2Z_{16}	Z_2^3

10	$Z_2^2Z_8+Z_3$	Z_8	Z_3	Z_2^2	4(32) $Z_2^4Z_3$ 20(64) $Z_2^3Z_{12}$ 52(64) $Z_2^3Z_8Z_3$ 12(16) $Z_2^3Z_3$	5(32) Z_{32} 21(64) Z_{64} 53(64) Z_{128} 13(16) Z_{16}	Z_2Z_3	Z_2^4
11	$Z_8^2+Z_{63}$	0	$.Z_8+Z_{63}$	3(32) Z_2^3 19(64) $Z_2^2Z_4$ 51(128) $Z_2^2Z_8$ 115(128) $Z_2^2Z_{16}$ 11(16) Z_2^2	4(32) $Z_8+Z_{32}+Z_{63}$ 20(64) $Z_8+Z_{64}+Z_{63}$ 52(128) $Z_8+Z_{128}+Z_{63}$ 116(128) $Z_4+Z_{256}+Z_{63}$ 12(16) $Z_8+Z_{16}+Z_{63}$	Z_2	$Z_2^2Z_8$	Z_2Z_8
12	0	0	2(32) Z_2^2 18(64) Z_2Z_4 ·50(128) Z_8Z_2 114(128) Z_2Z_{16} 10(16) Z_2	3(32) Z_{32} 19(64) Z_{64} 51(128) Z_{128} 115(128) Z_{256} 11(16) Z_{16}	Z_2	Z_2^2	Z_8	Z_8
13	Z_3	1(32) Z_2 17(64) Z_2Z_4 49(128) Z_8 Z_2 113(128) Z_2Z_{16} 9(16) Z_2	2(32) $Z_{32}+Z_3$ 18(64) $Z_{64}+Z_3$ 50(128) $Z_{128}+Z_3$ 114(128) $Z_{256}+Z_3$ 10(16) $Z_{16}+Z_3$	Z_2	$Z_2^2+Z_3$	Z_8	Z_8+Z_3	0

TABLE 2

Order of $\mathrm{im}(i_*\pi_j(s^n)\rightarrow\pi_j(V_{2n,n}))$
Top row is the name of the stem

$n(8)$	ι	η	η^2	ν	ν^2	σ	ϵ	$\bar{\nu}$	$\eta\sigma$	$\eta\epsilon$	$\eta\bar{\nu}$	$\eta^2\sigma$	μ	$\eta\mu$	ζ
0	∞	2	2	8	2	16	2	2	2	2	2	2	2	2	8
1	2	2	2	2	2	2	2	2	2	0	2	2	2	2	0
2	∞	2	0	4	2	16	0	2	2	0	0	0	2	0	4
3 115(128)	2	0	0	2	2	2	0	0	0	0	0	0	0	0	0 2
4	∞	2	2	8	0	16	2	2	2	2	0	2	2	2	8
5 53(64)	2	2	2	2	0	2	2	2	2	0 2	0	0 2	2	2	0
6 22(32)	∞	2	0	4	2	16	2	2	0 2	0	0	0	2	0	4
7 15	2	0	0	0	0	2 0	0	0	0	0	0	0	0	0	0

$=\binom{i}{j}h_{i+j}$. Hence the cohomology of the base space is given by h_{32n+} $=Sq^{i+1}h_{32n-1}$. We let $h_{32n-1}=h$.

1st level. Over the Steenrod algebra the basis for ker p^* is given by $\{Sq^7h, Sq^8h, Sq^{16}h\}$. Of these three only the first two can be spherical in the sense of [6]. Indeed using i: $V_{k+m,m}\rightarrow BSO(k)$ each class in $\pi_j(V_{k+m,m})$ represents a k-plane bundle over S^j which becomes trivial when summed with a trivial m-plane bundle. It is also easy to see that the bundle is a framed tangent bundle of S^j if and only if the cohomology map is nontrivial. Since the $15+32n$ sphere has only an eight field, $Sq^{16}h$ is not spherical. It is useful to kill it anyway but one has to be careful and identify the element at a later stage which is produced because of this.

2nd level. Consider the following fibering

$$K_1(Z, 32n+5)\times K_2(Z_2, 32n+6)\times K(Z_2, 32n+14) \xrightarrow{i} E^1 \xrightarrow{q} V_{32n+m,m+1}$$

with k-invariants Sq^7h, Sq^8h and $Sq^{16}h$.

TABLE 3

A modified Postnikov tower for $V_{k+m,m}$, $k\equiv 6(32)$, m large

Z	α_1							
Z_4	α_2	$\beta_1 Sq^1\alpha_2+Sq^2\alpha_1$						
0								
Z_4		$\beta_2 Sq^4\alpha_2+Sq^3\alpha_2$	$\gamma_1 Sq^1\beta_2+Sq^{2,1}\beta_1$					
0								
Z_8		$\beta_3 Sq^{4,1}\alpha_2$	$\gamma_2 Sq^1\beta_3+Sq^{2,3}\beta_1$	$\delta_1 Sq^1\gamma_2+Sq^{2,1}\gamma_1$				
Z_4		$\beta_4 Sq^{4,2}\alpha_2$	$\gamma_3 Sq^1\beta_4+(Sq^4+Sq^{3,1})\beta_2$ $+Sq^6\beta_1+Sq^2\beta_3$					
Z_{16}		$\beta_5 Sq^8\alpha_1$	$\gamma_4 Sq^1\beta_5+Sq^{2,3}\beta_2$	$\delta_2 Sq^1\gamma_4+Sq^{2,3}\gamma_1$	$\epsilon_1 Sq^1\delta_2+Sq^{2,1}\delta_1$			
			$\gamma_5 Sq^4\beta_3+Sq^{6,2}\beta_1$					
Z_4Z_2		$\beta_6 Sq^8\alpha_2$	$\gamma_6 Sq^1\beta_6+Sq^2\beta_5+Sq^4\beta_3$ $+Sq^3\beta_4+Sq^{4,2}\beta_2+Sq^8\beta_1$	$\delta_3 Sq^1\gamma_5+Sq^{2,1}\gamma_3$ $+Sq^4\gamma_2+Sq^6\gamma_1$				
Z_2Z_{32}	α_3	$\beta_7 Sq^1\alpha_3+Sq^{8,1}\alpha_2$	$\gamma_7 Sq^1\beta_7+Sq^{4,1}\beta_3+Sq^{7,2}\beta_1$	$\delta_4 Sq^1\gamma_7+Sq^{2,3}\gamma_2$	$\epsilon_2 Sq^1\delta_4+Sq^{2,3}\delta_1$	$\zeta_1 Sq^1\epsilon_2+Sq^{2,1}\epsilon_1$		
			$\gamma_8 Sq^4\beta_4+Sq^{6,3}\beta_1+Sq^7\beta_2$					
Z_2		$\beta_8 Sq^2\alpha_3+Sq^{8,2}\alpha_2$						
$Z_2^2Z_8$			$\gamma_9 Sq^2\beta_8+Sq^3\beta_7+Sq^{3,1}\beta_6$ $+Sq^{4,2}\beta_4+Sq^{9+8,1}\beta_2+Sq^{8,3}\beta_1$	$\delta_5 Sq^{2,1}\gamma_8+Sq^{3,1}\gamma_6$ $+Sq^4\gamma_5+Sq^{5+4,1}\gamma_4$ $+Sq^6\gamma_3+Sq^{8,1+6,3}\gamma_1$	$\epsilon_3 Sq^{3,1}\delta_3+Sq^{7+4,2,1}\delta_1$	$\zeta_2 Sq^1\epsilon_3+Sq^{4,1}\epsilon_1$	$\lambda_1 Sq^1\zeta_2+Sq^{2,1}\zeta_1$	
Z_8		$\beta_9 Sq^4\alpha_3+Sq^{8,4}\alpha_2$	$\gamma_{10} Sq^1\beta_9+Sq^{2,1+3}\beta_8+Sq^4\beta_7$ $+Sq^{4,1}\beta_6+Sq^{4,2,1}\beta_4$ $+Sq^{(8+7,1+6,2)}\beta_3$	$\delta_6 Sq^1\gamma_{10}+Sq^2\gamma_9+Sq^4(\gamma_7+\gamma_8)$ $+Sq^{3,1}\gamma_8+Sq^{5,2+4,2,1}\gamma_3$ $+Sq^8\gamma_2+Sq^{8,2+10+9,1}\gamma_1$				
Z_8						$\zeta_3 Sq^3\epsilon_3+Sq^{4,2,1}\epsilon_1$	$\lambda_2 Sq^1\zeta_3+Sq^{4,1}\zeta_1$	$N_1 Sq^1\lambda_2+Sq^{2,1}\lambda_1$

PROPOSITION. *A class $v \in H^j(E^1, Z_2)$ such that $v \notin \operatorname{im} q^*$ and $j \leqq 64n-3$ satisfies: $i^*v = \sum_{i=1}^{3} \beta_i \alpha_i$ where α_i is the fundamental class of K_i and β_i is an element of the Steenrod algebra such that $\beta_1 Sq^7 + \beta_2 Sq^8 + \beta_3 Sq^{16}$, as an element in the Steenrod algebra, has only classes of length 2 or more in its Cartan basis representation.*

Using this representation of $H^*(E^1)$ it is now just a lengthy but straight forward computation to verify that the classes in Table 3, column 2 do form a basis over the Steenrod algebra for $H^j(E^1)$ if $32n+7 \leqq j \leqq 32n+21$.

3rd level. Consider the fibering

$$\prod_{i=1}^{9} K_i(Z_2, n_i) \to E^2 \to E^1$$

with k-invariants given by Table 3. We use β_i to represent also the fundamental class of K_i. The value of n_i can be inferred from the table. Consider the diagram

$$\begin{array}{ccccc} H^*(E^2) & \xrightarrow{i_2^*} & H^*(\pi K_i(Z_2, n_i)) & \xrightarrow{\delta^*} & H^*(E^2, \pi K_i) \\ & & & \searrow^{\tau} & \uparrow \simeq \\ & & & & H^*(E^1) \underset{i_1^*}{\to} H^*(K_1 \times K_2 \times K_3) \end{array}$$

PROPOSITION. *A class $v \in H^j(E^2)$, $7 \leqq j - 32n \leqq 21$, is defined uniquely by a sum $\sum_{i=1}^{9} a_i \beta_i$ satisfying:*

(1) $i_2^* v = \sum_{i=1}^{9} a_i \beta_i$ and
(2) $\sum a_i (i_1^* \tau \beta_i) = 0$.

This is a special case of 3.3.4 of [6].

Using this proposition the cohomology of E^2 in the interesting range can be computed. Another lengthy computation shows that column 3 of Table 3 is a basis over the Steenrod algebra for $H^j(E^2)$, $7 \leqq j - 32n \leqq 21$.

4th and higher levels. The computations are made as in the third level, using 3.3.4 of [6]. Nothing unusual happens. The class corresponding to $\gamma_6 + \delta_5$ is the extraneous class produced by killing $Sq^{16}h$. This follows from Toda [8]. It is amusing to note that the formula of Adams [0]

$$Sq^{16} = \sum a_{i,j,3} \phi_{i,j}$$

with coefficients, for example, $a_{3,3,3} = Sq^1$ and $a_{1,3,3} = Sq^7 + Sq^4 Sq^2 Sq^1$, essentially given by γ_6.

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