PIECEWISE LINEAR NORMAL MICRO-BUNDLES

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The object of this paper is to gain some information about the unstable piecewise linear groups. The tool that we use for this purpose is the s-cobordism theorem (which has been established for piecewise linear manifolds by J. Stallings [9] and D. Barden [1]). All manifolds and micro-bundles in this paper are piecewise linear, unless otherwise specified.

THEOREM 1. *Let M^m be a compact manifold such that $\pi_1(\partial M)\cong\pi_1 M$ by inclusion, and let $f\colon K^k\to M^m$ be a simple homotopy equivalence of a finite simplicial k-complex with M. Then if $m\geqq 6$, $m\geqq 2k+1$, there is a compact manifold L such that $\pi_1(\partial L)\cong\pi_1 L$, and $L\times I\cong M$. If $m\geqq 7$, $m\geqq 2k+2$, L is uniquely determined.*

PROOF. We first observe that the pair $(M, \partial M)$ is $(m-k-1)$-connected. Indeed, since inclusion induces an isomorphism of fundamental groups, we can use the relative Hurewicz theorem to compute the first nonvanishing relative homotopy group: $\pi_i(M, \partial M)\cong\pi_i(\tilde{M}, \partial\tilde{M})$ (where $\tilde{M}$ denotes the universal cover), $\pi_i(\tilde{M}, \partial\tilde{M})\cong H_i(\tilde{M}, \partial\tilde{M})\cong H_c^{m-i}(\tilde{M})$, by duality, where c denotes compact cohomology, and $H_c^{m-i}(\tilde{M})\cong H_c^{m-i}(\tilde{K})$ vanishes for $i<m-k$.

It follows that for $m\geqq 2k+1$, f is homotopic to a map $g\colon K\to\partial M$. If, now $m\geqq 2k+2$ we can move g into general position (see e.g. [11, Chapter 6, Theorem 18]) and so suppose it an imbedding. Take a regular neighbourhood L of $g(K)$ in ∂M. Then L is a manifold, and the inclusion $L\subset M$ is a simple homotopy equivalence.

If $m=2k+1$, g will in general have singularities, transverse self-intersections of k-simplexes of K. For each such selfintersection $Q=g(P_1)=g(P_2)$, we join P_1 to P_2 by a path α in K such that $g(\alpha)$ is a nullhomotopic loop (since $g_*\colon\pi_1(K)\to\pi_1(\partial M)$ is onto, this is possible). As $k\geqq 3$, we can now map a disc D^2 into ∂M, with its interior imbedded, and meeting $g(K)$ only in its boundary, which is attached along $g(\alpha)$. Proceeding thus for each selfintersection Q, we obtain an imbedding of a complex K' simply homotopy-equivalent to K; we can then take a regular neighbourhood to obtain L, as above. Note in either case that as regular neighbourhood of a subcomplex K' of codimension $\geqq 3$, L has the property $\pi_1(\partial L)\cong\pi_1 L$, for ∂L is a deformation retract of $L-K'$.

Take a collar neighbourhood $\partial L\times I$ of L in ∂M (this is possible since L is a submanifold); let L^1 be the closure of the complement of $L\cup(\partial L\times I)$ in ∂M. We regard M as a cobordism of L and L^1: along the 'edge', $\partial L\times I$ is a product cobordism of ∂L and ∂L^1. Also, the inclusion of L in M is a simple homotopy equivalence. To show that M is an s-cobordism, it remains only to check that the inclusion $L^1\subset M$ induces an isomorphism of π_1. Now the complement of L^1 in ∂M is a regular neighbourhood of a k-complex, which has codimension $\geqq 3$, so $\pi_1(L^1)\cong\pi_1(\partial M)$; and by hypothesis, $\pi_1(\partial M)\cong\pi_1(M)$. Hence M is an s-cobordism which along the edge is a product cobordism; by the s-cobordism theorem, M is a product: $M\cong L\times I$.

For the proof of uniqueness, we first show that L is in any case the regular neighbourhood of a k-complex, given $m\geqq k+4$, $m\geqq 6$. For by assumption $\pi_1(\partial L)=\pi_1(L)$; now, as in the proof of existence, $(L, \partial L)$ is $(m-k-2)$-connected. By [10, Theorem 5.5], if $k\geqq 2$, L has a handle decomposition based on ∂L with no i-handles for $i\leqq m-k-2$; the dual decomposition has no j-handles for $j>k$, and so L collapses onto a k-dimensional spine. If $k=1$, since $m\geqq 6$ we can imbed K in L by a simple homotopy equivalence and take a regular neighbourhood L' of the image; by [10, Theorem 6,4] (a variant of the s-cobordism theorem) L is diffeomorphic to L'. We observe that the arguments of [10] can be justified for PL-manifolds by using results from [1] or [9]; we could also use a PL version of the nonstable neighbourhood theorem of Mazur [7, p. 54].

Suppose then $M=L_1\times I\cong L_2\times I$, and consider the image of $L_2\times 0$ in ∂M. Since ∂M has dimension $\geqq 2k+1$, we can deform this to be disjoint from $L_1\times 1$, and then a further deformation puts it in the interior of $L_1\times 0$. Write H for the closure of $(L_1\times 0)-(L_2\times 0)$; we assert that H is an s-cobordism, and hence a product $\partial L_2\times I$, so that L_1 is homeomorphic to L_2. This can be proved algebraically, or we can use a direct argument by cancellation of handles: for details see Wall [10, Theorem 6.4].

We now consider piecewise linear micro-bundles. The basic information on these is contained in Milnor [8]. We write ϵ^r for the trivial micro-bundle with fibre $\mathbf{R}^r$.

COROLLARY 1.1. *For any micro-bundle* ξ^r *over* K^k, *we can write* $\xi^r+\epsilon^{2k}\cong\epsilon^r+\eta^{2k}$ *for a suitable micro-bundle* η^{2k} *of fibre dimension* $2k$. (*If* $k=2$, *replace* $2k$ *by* 5.)

PROOF. First suppose $k\geqq 3$. Then, as in the theorem, we can imbed some complex simple homotopy-equivalent to K in $\mathbf{R}^{2k}$; thicken it, and call the result L. The tangent micro-bundle of L is ϵ^{2k}. Take the total space M^1 of the micro-bundle induced over L by ξ^r, and let M be a regular neighbourhood of L in M^1: this has tangent micro-bundle $\xi^r+\epsilon^{2k}$. Now by iterating the theorem, we can write $M=N^{2k}\times D^r$, so if η^{2k} is the tangent micro-bundle of N^{2k}, the result follows.

If $k=2$, we replace $\boldsymbol{R}^4$ by $\boldsymbol{R}^5$, so L has dimension 5. The argument concludes as before.

COROLLARY 1.2. *Suppose ξ^r and η^r are stably equivalent micro-bundles over K^k. Then $\xi^r+\epsilon^{2k}\cong\eta^r+\epsilon^{2k}$. (If $k=2$, replace $2k$ by 5.)*

PROOF. Construct L as above; take regular neighbourhoods X and Y of L in the micro-bundles induced over L by ξ and η. Since ξ and η are stably equivalent, for some s, $X\times D^s\cong Y\times D^s$. Since X and Y have dimension $2k+r\geqq 2k+1$, it follows by iterating the uniqueness part of Theorem 1 that X and Y are PL-homeomorphic. Hence their tangent micro-bundles $\xi^r+\epsilon^{2k}$, $\eta^r+\epsilon^{2k}$ are equivalent.

REMARK 1. To classify micro-bundles over a 1-complex, it is sufficient to be able to do it over a circle; for this we only need $\pi_0(\mathrm{PL}_m)$, which is well known to be $\boldsymbol{Z}_2$. Thus if $k=1$, we have $\xi^r=\epsilon^{r-1}+\eta^1$, and stably equivalent micro-bundles are equivalent.

THEOREM 2. *Suppose K^k a compact unbounded piecewise linear submanifold of M^m. Then if $m\geqq 3k$, K^k has a piecewise linear normal micro-bundle in M^m.*

PROOF. First assume $k\geqq 3$. According to Milnor [8, Theorem 4], for some n, K^k has a normal micro-bundle ξ^r in $M^m\times\boldsymbol{R}^n$. By the above corollary, write $\xi^r+\epsilon^{2k}=\epsilon^r+\eta^{2k}$; let N_1 be a regular neighbourhood of K in the total space of $\eta+\epsilon^{m-3k}$, N_2 a regular neighbourhood of K in M. Then $N_1\times D^n$, $N_2\times D^n$ are both regular neighbourhoods of K in $M\times\boldsymbol{R}^n$, hence are PL-homeomorphic.

By Theorem 1, if $m\geqq 6$, N_1 and N_2 are PL-homeomorphic. We assert that there is even a PL-homeomorphism inducing the identity on the subcomplex K. Granted this, K has a normal micro-bundle in N_1, hence also in N_2, and so in M.

Write $i_1\colon K\to N_1$, $i_2\colon K\to N_2$ for the inclusions, and $f\colon N_1\to N_2$ for the PL-homeomorphism constructed above. Then, by the construction of f, $fi_1\cong i_2$. Since dim $N_2=3k\geqq 2k+2$, homotopic imbeddings are isotopic. By the covering isotopy theorem of Hudson and Zeeman [5], since (as $k\geqq 2$) the codimension is $\geqq 3$, we can cover the isotopy of K in N by an isotopy h_t of N. Hence $h_1fi_1=i_2$. The homeomorphism h_1f now has the required properties.

In low dimensions we can use a different argument. For if $k\leqq 7$, it follows from smoothing theory (see e.g. [3]) that N_2 and K admit compatible differential structures; if also $2m \geqq 3k+3$, by [2, Theorem 2a] the imbedding of K in N_2 can be approximated by a differentiable imbedding; if finally $2m \geqq 3k+4$ these two imbeddings, being homotopic, are PL-isotopic by a theorem of Hudson [4]. Hence M^m can be regarded as a smooth manifold with K^k as smooth submanifold; as such it has a normal vector bundle and hence a normal PL-microbundle, according to [6, Theorem 3.2].

ADDENDUM TO THEOREM 2. *The result also holds if* $k \leqq 7$, $2m \geqq 3k+4$.

This includes those cases of the theorem which were not covered by our first argument.

REMARK 2. The necessity of suspending ξ in the corollaries to Theorem 1—as also the lack of a uniqueness clause in Theorem 2—all stem from our inability, given a complex K and PL-micro-bundle ξ^r over K, to construct a manifold M^r and homotopy equivalence $h: M^r \rightarrow K$, such that $h^*\xi$ is equivalent to the tangent micro-bundle of M. (However large r is, we cannot yet do this.)

Added in proof. Haefliger and the author have now proved a stability theorem for PL-micro-bundles fully analogous to the stability properties of vector bundles, and deduced that Theorem 2 holds for $m \geqq 2k$.

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