SEMI-REGULAR MAXIMAL ABELIAN SUBALGEBRAS IN HYPERFINITE FACTORS[1]

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A factor is a ring of operators whose center consists only of scalar multiples of the identity. Murray and von Neumann have defined various kinds of factors, calling a continuous factor with finite trace a type II_1 factor [3], [4]. Dixmier began the detailed study of maximal abelian subalgebras of type II_1 factors. He defined regular, semi-regular but not regular, and singular maximal abelian subalgebras, and showed that at least one of each type exists [2]. His II_1 factors turn out to be hyperfinite in algebraic type. The factors we consider are also hyperfinite. In this note we discuss their semi-regular subalgebras, and present an isomorphism invariant which allows us to obtain new existence results.

Let $\mathfrak{A}$ be a hyperfinite factor, $\boldsymbol{R}$ a maximal abelian subalgebra of $\mathfrak{A}$. For any subring $\boldsymbol{D}$ of $\mathfrak{A}$, $N(\boldsymbol{D})$ is the ring generated by all unitaries which leave $\boldsymbol{D}$ invariant, and $N^k(\boldsymbol{D}) = N[N^{k-1}(\boldsymbol{D})]$. In particular, we let $N(\boldsymbol{R}) = \boldsymbol{P}$. $\boldsymbol{R}$ is semi-regular but not regular iff $\boldsymbol{P}$ is a factor not equal to $\mathfrak{A}$. In [5] we defined an isomorphism invariant for such subalgebras, which we called *length.* If $\boldsymbol{R} \subset \boldsymbol{P} \subset N(\boldsymbol{P}) \subset \cdots \subset N^L(\boldsymbol{P}) = \mathfrak{A}$, (when $\boldsymbol{R} \neq \boldsymbol{P} \neq N(\boldsymbol{P}) \neq \cdots \neq N^L(\boldsymbol{P})$) then L is the length of $\boldsymbol{R}$. By constructing a semi-regular subalgebra $\boldsymbol{R}$ of every length $L = 1, 2, 3, \cdots$, we obtained an infinite sequence of subalgebras which could not be pairwise connected by *-automorphisms of $\mathfrak{A}$.

Another possible invariant is *product type.* Suppose $\boldsymbol{R}$ has length L. Then $\boldsymbol{R}$ is of product type α, $0 \leqq \alpha \leqq L$, iff the following holds: For every t, $1 \leqq t \leqq \alpha$, there exist S_1 and S_2 in $N^{t-1}(\boldsymbol{P})^{\perp} \cap N^t(\boldsymbol{P})$ such that the product $S_1 S_2 \neq 0$ is in $N^{t-1}(\boldsymbol{P})^{\perp} \cap N^t(\boldsymbol{P})$. But for s such that $\alpha \leqq s \leqq L$, every T_1 and T_2 in $N^{s-1}(\boldsymbol{P})^{\perp} \cap N^s(\boldsymbol{P})$ have their product $T_1 T_2$ in $N^{s-1}(\boldsymbol{P})$. (Taking of orthogonal complements is meaningful, for within a II_1 factor, the weak, strong, and Hilbert space (metric) closures of a subalgebra all coincide [4]. The metric topology is based on the norm derived from the scalar product $(A, B) = \mathrm{Tr}(B^* A)$ for A, B in $\mathfrak{A}$.)

THEOREM 1. *Suppose $\boldsymbol{R}$ and $\boldsymbol{R}'$ are semi-regular but not regular subalgebras of $\mathfrak{A}$, and $\boldsymbol{R}$ has product type α, while $\boldsymbol{R}'$ has product type*

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*α', $\alpha \neq \alpha'$. Then there does not exist a *-automorphism Θ of $\mathfrak{A}$ such that $\Theta(\boldsymbol{R}') = \boldsymbol{R}$.*

PROOF. We can assume $\alpha' > \alpha$, so that $\alpha' \geqq \alpha+1$. Letting $t = \alpha+1$ in the definition of product type, we know we can choose S_1, S_2 in $N^\alpha(\boldsymbol{P}')^\perp \cap N^{\alpha+1}(\boldsymbol{P}')$ such that $S_1S_2 = S_3 \neq 0$ is in this set also. Suppose there exists Θ such that $\Theta(\boldsymbol{R}') = \boldsymbol{R}$. By a standard argument, it follows that $\Theta[N(\boldsymbol{R}')] = N(\boldsymbol{R})$ or $N(\boldsymbol{P}') = \boldsymbol{P}$, and inductively, $\Theta[N^\alpha(\boldsymbol{P}')] = N^\alpha(\boldsymbol{P})$ and $\Theta[N^{\alpha+1}(\boldsymbol{P}')] = N^{\alpha+1}(\boldsymbol{P})$. Let $\Theta(S_i) = T_i$, so that $T_i \in N^{\alpha+1}(\boldsymbol{P})$ for $i = 1, 2$, or 3. Now if $A \in N^\alpha(\boldsymbol{P}')$, then $(S_i, A) = 0 = \mathrm{Tr}(A^* S_i) = \mathrm{Tr}[\Theta(A^* S_i)]$ (since the trace function is unique) $= \mathrm{Tr}[\Theta(A)^* \Theta(S_i)] = (\Theta(S_i), \Theta(A)) = (T_i, \Theta(A)) = 0$. As A ranges over $N^\alpha(\boldsymbol{P}')$, $\Theta(A)$ takes on all values in $N^\alpha(\boldsymbol{P})$, so we must have T_i in $N^\alpha(\boldsymbol{P})^\perp$. Thus T_i is in $N^\alpha(\boldsymbol{P})^\perp \cap N^{\alpha+1}(\boldsymbol{P})$ for $i = 1, 2$, or 3.

Now $\alpha+1 > \alpha$, so letting $s = \alpha+1$ and considering the product type of $\boldsymbol{R}$, it follows that T_1T_2 is in $N^\alpha(\boldsymbol{P})$. But $T_1T_2 = \Theta(S_1)\Theta(S_2) = \Theta(S_1S_2) = \Theta(S_3) = T_3$. Since T_3 is in $N^\alpha(\boldsymbol{P})^\perp$, this leads to a contradiction. ($T_3 \neq 0$ since $S_3 \neq 0$ and Θ is an automorphism.) Therefore we cannot have $\Theta(\boldsymbol{R}') = \boldsymbol{R}$.

THEOREM 2. *There are $(L+1)$ semi-regular subalgebras of length L which cannot be pairwise connected by *-automorphisms of $\mathfrak{A}$. Specifically, these have product types $\alpha = 0, 1, 2, \cdots, L$.*

We give an indication of the proof, which is constructive and depends on the results of [5]. For each $n = 1, 2, 3, \cdots$, the matrix units (of all the 2^p by 2^p matrix algebras, where p is an odd multiple of n) are divided into n orthogonal sets. These are called $\mathfrak{N}_0, \mathfrak{N}_1, \cdots, \mathfrak{N}_n$, and the set $\mathcal{C}_k = \bigcup_{j=0}^{k} \mathfrak{N}_j$. The ring $R(\mathcal{C}_k)$ is defined as the weak closure of the algebra generated by matrix units in $\mathcal{C}_k$. Then for each n and for $0 \leqq \alpha \leqq n-\alpha$, we construct $\boldsymbol{R}_n(\alpha)$, a semi-regular subalgebra. The chain for $\boldsymbol{R}_n(\alpha)$ is such that $N^t(\boldsymbol{P}_n(\alpha)) = R(\mathcal{C}_{2t})$ for $0 \leqq t \leqq \alpha$ and $N^s(\boldsymbol{P}_n(\alpha)) = R(\mathcal{C}_{\alpha+s})$ for $\alpha \leqq s \leqq n-\alpha$. Since $N^{n-\alpha}(\boldsymbol{P}_n(\alpha)) = R(\mathcal{C}_n) = \mathfrak{A}$, we have $L = n-\alpha$.

But these properties are sufficient to show that $\boldsymbol{R}_n(\alpha)$ has product type α. So for any $L = 1, 2, 3, \cdots$, we can take $\alpha = 0, 1, \cdots, L$, and $n = \alpha+L$. Then, by Theorem 1, there does not exist an *-automorphism Θ of $\mathfrak{A}$ such that $\Theta(\boldsymbol{R}_{\alpha+L}(\alpha)) = \boldsymbol{R}_{\alpha'+L}(\alpha')$ when $\alpha \neq \alpha'$.

A generalization of the concept of product type permits one to construct 2^L nonisomorphic semi-regular maximal abelian subalgebras of every length L. However, the construction becomes extremely involved.

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PURE SUBGROUPS HAVING PRESCRIBED SOCLES

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Let $B=\sum B_n$ be a direct sum of cyclic groups where, for each positive integer n, $B_n=\sum C(p^n)$ is zero or homogeneous of degree p^n where p is a fixed prime. Denote by $\bar{B}$ the torsion completion of B in the p-adic topology. Following established terminology [1], we refer to $\bar{B}$ as the closed primary groups with basic subgroup B. A primary group G is said to be pure-complete if each subsocle of G supports a pure subgroup of G. A semi-complete group was defined by Kolettis in [6] to be a primary group which is the direct sum of a closed group and a direct sum of cyclic groups.

For a particular B, I exhibited in [3] nonisomorphic pure subgroups H and K of $\bar{B}$ having the same socle. Using this example, Megibben [7] was the first to show the existence of a primary group without elements of infinite height which is not pure-complete. We mention that each semi-complete group is pure-complete [4]. The purpose of this note is to announce the following theorem and corollaries; proofs will appear in another paper.

THEOREM. *Suppose that B is unbounded and countable and that S is any proper dense subsocle of $\bar{B}$ such that $|S|=2^{\aleph_0}$. Then S supports more than $2^{\aleph_0}$ pure subgroups of $\bar{B}$ which are isomorphically distinct.*

The theorem has the following implications.

COROLLARY 1. *Suppose that B is unbounded and countable and that*