ON CHERN CLASSES OF REPRESENTATIONS OF FINITE GROUPS

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Let $R(G)$ denote the complex representation ring of a finite group G. Any complex representation ρ of G has invariants $c_n(\rho) \in H^{2n}(G; \boldsymbol{Z})$, the *Chern classes* of ρ (Atiyah [1]).

If H is a subgroup of G, there is the *induced representation* homomorphism

$$i_!: R(H) \to R(G)$$

(cf. [8], say). Atiyah [1] posed the problem of relating the Chern classes of $i_!\lambda$ with those of λ, for any representation λ of H. The purpose of this note is to announce the proof of a conjecture of J. F. Adams which gives some information in this direction; the main idea of the proof was suggested to me by Professor Adams, and is believed to emanate essentially from Professor Atiyah. I would like to thank Professor Adams sincerely for his help, and to acknowledge the helpfulness of Professor Atiyah and Professor M. G. Barratt.

The result to be proved involves the *transfer* homomorphism

$$i_!: H^*(H; \boldsymbol{Z}) \to H^*(G; \boldsymbol{Z})$$

(cf. [6], [8]), and certain linear maps

$$\mathrm{Ch}_k: R(L) \to H^{2k}(L; \boldsymbol{Z})$$

defined, for any finite group L, in terms of the Chern classes as follows:

Let $Q^k(\sigma_1, \cdots, \sigma_n)$ be the polynomial defined by expressing the symmetric polynomial $x_1^k + \cdots + x_n^k$ in indeterminates $x_1, \cdots, x_n$ in terms of the elementary symmetric polynomials $\sigma_i(x_1, \cdots, x_n)$. If $\rho: L \to U(n)$ is a representation of L of degree n, then

$$\mathrm{Ch}_k(\rho) = Q^k(c_1(\rho), \cdots, c_n(\rho)) \in H^{2k}(L; \boldsymbol{Z}).$$

THEOREM 1. *Given any positive integer k, there exists an integer N_k with the following property:*

If H is an arbitrary subgroup of an arbitrary finite group G, then the following diagram of homomorphisms commutes:

$$\begin{array}{ccc} R(H) & \xrightarrow{i_!} & R(G) \\ N_k\mathrm{Ch}_k \downarrow & & \downarrow N_k\mathrm{Ch}_k \\ H^{2k}(H;\boldsymbol{Z}) & \xrightarrow[i_!]{} & H^{2k}(G;\boldsymbol{Z}). \end{array}$$

If N_k denotes the least positive integer with this property, and G' is a group of order prime to N_k, it follows that

$$\mathrm{Ch}_k i_! = i_! \mathrm{Ch}_k$$

for all monomorphisms $i: H \to G'$.

The case $k=0$ of this theorem is trivial. The case $k=1$ follows from a previously unpublished lemma of J. F. Adams below, while the case $k>1$ is dealt with, along lines suggested by Professor Adams, by means of a certain "Riemann-Roch type" lemma (Lemma 3).

LEMMA 2 (J. F. ADAMS). *If $\lambda \in R(H)$, then*

$$c_1(i_!\lambda) = i_! c_1(\lambda) + (\deg \lambda) c_1(i_! 1),$$

where 1 is the trivial representation of H.

This lemma is proved by exploiting a very explicit algebraic description of the first Chern class (cf. [1]), and standard explicit algebraic descriptions of the maps $i_!$ (cf. [8], say). The proof shows that $c_1(i_!1)$ always has order dividing 2. Further, the example of $\{1\} \subset \boldsymbol{Z}_2$ shows that 2 is the least positive integer N_1 such that $N_1(\mathrm{Ch}_1 i_! - i_! \mathrm{Ch}_1) \equiv 0$.

The general case of the theorem is deduced from the following lemma:

LEMMA 3. *Let $f: X \to Y$ be a covering map of compact almost-complex manifolds. Then the following diagram commutes:*

$$\begin{array}{ccc} K_C^*(X) & \xrightarrow{f_!} & K_C^*(Y) \\ M_k\mathrm{Ch}_k \downarrow & & \downarrow M_k\mathrm{Ch}_k, \\ H^*(X;\boldsymbol{Z}) & \xrightarrow[f_!]{} & H^*(Y;\boldsymbol{Z}) \end{array}$$

where $M_k = \prod_{r=1}^{k} (n+r)!/r!$, $2n = \dim_R X$, Y, and the maps $f_!$ are those given by using Thom isomorphisms defined by normal bundles to X and Y.

OUTLINE OF PROOF. First suppose that $f: X \to Y$ is an arbitrary map of almost-complex manifolds. Let ϕ_H, ϕ_K denote Thom isomorphisms in integral cohomology and in K-theory defined by normal bundles to a given almost-complex manifold W. Write

$$B_k(W) = \overset{-1}{\phi_H} \operatorname{Ch}_k \phi_K(1).$$

By methods similar to some used in [5] one obtains the formula:

$$\sum_{r=0}^{k} \binom{k}{r} [B_r(Y).\operatorname{Ch}_{k-r} f_! x - f_!(B_r(X).\operatorname{Ch}_{k-r} x)] = 0 \quad [x \in K_C^*(X)].$$

In the case that f is a finite covering, $B_r(X) = f^* B_r(Y)$. Further, if $2n = \dim_R X$, Y, the Bott results on $K_C(S^{2n})$ imply that $B_n(Y) = n!$. Hence, in this case, the formula reduces to the equation

$$\frac{k!}{(k-n)!} (\operatorname{Ch}_{k-n} f_! x - f_! \operatorname{Ch}_{k-n} x)$$
$$= - \sum_{r=n+1}^{k} \binom{k}{r} B_r(Y)[\operatorname{Ch}_{k-r} f_! x - f_! \operatorname{Ch}_{k-r} x] \quad [x \in K_C^*(X)].$$

The required result now follows by induction.

The following lemma is an immediate consequence of a result of J.-P. Serre (quoted in [2]).

LEMMA 4. *Let H be a subgroup of a finite group G. For any integer $n > 2$, there exists a covering map $p: X_H \to X_G$ of projective complex algebraic manifolds both of (real) dimension $2(n+1)$ and such that X_H, X_G have the same homotopy n-type as products of Eilenberg-MacLane spaces $K(\mathbf{Z}, 2) \times K(H, 1)$, $K(\mathbf{Z}, 2) \times K(G, 1)$, respectively.*

The required theorem is now proved in dimension $2k$ by considering a covering map of this type when $n = 2k$, and applying Lemma 3. (A step of this kind was suggested in a letter by Professor Atiyah.) In that case it remains to be shown that the maps $p_!$ coincide with the algebraically-defined transfer maps. This is accomplished with the aid of results which appear in [1], [7], [3] and [5]; these results reduce the problem finally to that of comparing the K-theory transfer map with that defined by Grothendieck (cf. [4]) in terms of sheaves. (This is done in a final lemma.)

REFERENCES

1. M. F. Atiyah, *Characters and cohomology of finite groups*, Inst. Hautes Études Sci. Publ. Math. No. 9, Paris, 1961.

2. M. F. Atiyah and F. Hirzebruch, *Analytic cycles on complex manifolds*, Topology **1** (1961), 25–45.

3. ———, *The Riemann-Roch theorem for analytic embeddings*, Topology **1** (1962), 151–166.

4. A. Borel and J.-P. Serre, *Le théorème de Riemann-Roch (d'après Grothendieck)*, Bull. Soc. Math. France **86** (1958), 97–136.

5. E. Dyer, *Relations between cohomology theories*, Aarhus Colloquium on Algebraic Topology, 1962; pp. 89–93.

6. B. Eckmann, *Cohomology of groups and transfer*, Ann. of Math. (2) **58** (1953), 481–493.

7. ———, *On complexes with operators*, Proc. Nat. Acad. Sci. U.S.A. **39** (1953), 35.

8. M. Hall, Jr., *Theory of groups*, Macmillan, New York, 1959.

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ISOMORPHIC COMPLEXES

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In this paper we show that if K and L are n-complexes, then K and L are isomorphic iff the 1-sections of the first derived complexes of K and L are isomorphic. This provides a low-dimensional method for establishing the isomorphism (homeomorphism) of complexes (polyhedra).

Throughout, s_p will denote a (rectilinear) p-simplex with vertices $a^0, a^1, \cdots, a^p$; K will denote a (finite geometric) complex with n-section K^n and first derived complex K'. The *closed star* of a vertex a of K, $\mathrm{st}(a)$, is the set of simplexes of K having a as a face and all their faces. For more details see [2].

DEFINITION 1. An n-complex K is *full* provided, for any subcomplex L of K which is isomorphic to s_p^1, $2 \leqq p \leqq n$, L^0 spans a p-simplex of K.

THEOREM 1. *Suppose K and L are full n-complexes. Then K and L are isomorphic iff K^1 and L^1 are isomorphic.*

PROOF. We need only consider the case when K^1 and L^1 are isomorphic. Let $v: K^1 \to L^1$ be an admissible vertex transformation of K^1 onto L^1 with an admissible inverse. Then a^0, a^1 span a 1-simplex of K iff $v(a^0), v(a^1)$ span a 1-simplex of L. Furthermore, for any p, $2 \leqq p \leqq n$, if $a^0, a^1, \cdots, a^p$ span a p-simplex s_p of K, then $v[s_p^1]$ is isomorphic to s_p^1. So, using the fullness of L, we get that $(v[s_p^1])^0$