DIRECT FACTORS OF (AL)-SPACES[1]

BY DAVID W. DEAN

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Let E be a closed sublattice of the (AL)-space L [1, pp. 107–110]. The purpose of this note is to prove that there is a projection of L onto E having norm one. In particular then E is a direct factor of L. To show this we prove auxiliary theorems that the conjugate space E' of E may be "lifted" to L' (see Theorem 2 below) and that there is a projection of $E''=(E')'$ onto $q(E)$, the natural embedding of E in E'', whose norm is one.

The space L' is isometric and lattice isomorphic to a space $C(H)$ of functions continuous on a compact, extremally disconnected Hausdorff space H [4, Theorems 6.3, 6.9, Corollary 6.2]. Then L'' is isometric and lattice isomorphic to the space $R(H)$ of regular measures on H [3, p. 265]. If $x'\in L'$, $x''\in L''$ correspond to $f\in C(H)$, $\nu\in R(H)$, then $x''(x')=\int_H f d\nu=(\text{def. } \nu(f))$. If $\nu\in R(H)$ and $\nu(N)=0$ for each nowhere dense set N then ν is a normal measure. The support A_ν [2, pp. 2, 8, Proposition 3] of such a measure is both open and closed. Let $N(H)$ denote the subspace of normal measures.

THEOREM 1. *The representation of L'' as $R(H)$ maps $q(L)$ onto the space $N(H)$ of normal measures on H. Moreover $\bigcup\{A_\nu \mid \nu\in N(H)\}$ is dense in H (so that H is hyperstonean [2]).*

PROOF. Let $\nu\geqq 0$ correspond to qx for x in L. Let N be a closed nowhere dense set. We prove first that $\nu(N)=0$. Let F be the subset of functions f in $C(H)$ for which $\|f\|=1$, $f\geqq 0$, $f(h)=1$ if $h\in N$. Then F is directed by $\geqq$. This directed set then converges at each such ν to $\inf\{\nu(f)\mid f\in F\}$. Thus F converges on the representation of L in $R(H)$. The directed set of x' in L' corresponding to F then converges pointwise on L to an element y' in L'. If y' corresponds to g in $C(H)$ we have $\nu(g)=\inf\{\nu(f)\mid f\in F\}$ and clearly $g=\inf\{f\in F\}$. Since N is nowhere dense, $g=0$. Thus $\inf\{\nu(f)\mid f\in F\}=0$ so that $\nu(N)=0$. Thus ν is a normal measure.

To prove the second part let A be open and closed in H. For some $\nu>0$ corresponding to qx, $x\in L$, we have $\nu(\chi_A)>0$, where χ_A is the characteristic function of A. Thus A meets the support of ν. Hence H is hyperstonean. The theorem follows immediately from a result of Dixmier [2, p. 21, the corollary and its proof].

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THEOREM 2. *Let i be the inclusion mapping $E \rightarrow L$ and let $i'\colon L' \rightarrow E'$ denote the conjugate mapping. There is a positive isometry $T\colon E' \rightarrow L'$ such that $i'T$ is the identity on E'. Thus Tx' is an extension of x' to L for each x' in E'.*

PROOF. The space E is itself an (AL)-space and so it is isometric and lattice isomorphic to a space $N(K)$, with conjugate $C(K)$, and second conjugate $R(K)$, as above. We shall then, suppressing the representation mappings and their inverses, write $N(K) \rightarrow^{i} N(H)$, $C(H) \rightarrow^{i'} C(K)$, and seek a positive isometry $T\colon C(K) \rightarrow C(H)$ with the property that $i'T$ is the identity on $C(K)$.

Let χ_A denote the characteristic function of the set A. Let $\mathfrak{A}$ be the collection of open and closed subsets of H such that $i'\chi_A = 0$ and let $G = \mathrm{Cl}(\bigcup\{A \in \mathfrak{A}\})$. Then G is open and closed and we now show that $i'\chi_G = 0$. It is enough to show $\nu(G) = 0$ if $\nu \geqq 0$ is in $i(N(K))$ since $i(N(K))$ is a sublattice of $N(H)$. Since ν is normal, $\nu(G) = \nu(\bigcup\{A \in \mathfrak{A}\})$ as $G - \bigcup\{A \in \mathfrak{A}\}$ is nowhere dense. If $C \subset \bigcup\{A \in \mathfrak{A}\}$ is closed, a finite number of A's in $\mathfrak{A}$ cover C, so that $\nu(C) = 0$. Since ν is regular, $\nu(\bigcup\{A \in \mathfrak{A}\}) = 0$.

Now let e be an extreme point of the unit ball of $C(K)$ (so that e takes only the values 1 and -1). As a functional on $N(K)$ e has an extension to $C(H)$ which is an extreme point of the unit ball of $C(H)$. This follows since the set of norm one extensions of e is a compact convex set in the w'-topology of $C(H)$. This set then has an extreme point and such a point is also an extreme point of the unit ball of $C(H)$. Let f agree with such an extension off G and have value 0 on G. Then f is an extension of e to $C(H)$ and takes only the values 1 and -1 off G. Then f has the following properties. (a) $i'f^+ = e^+$ (so $i'f^- = e^-$), (b) f is unique. To show (a) note that $\chi_K \geqq i'f^+ \geqq 0$, $\chi_K \geqq i'f^- \geqq 0$ and $i'f = i'(f^+ - f^-) = e$ (i' is a positive norm one mapping). Thus if $e(k) = 1$ then $i'f^+(k) = 1$ and if $e(k) = -1$ then $i'f^-(k) = 1$. Hence (a). To show (b) let g be another such f. By (a) $i'g^+ = e^+$. Let $A' = \{h \mid g(h) = 1,\ f(h) = -1\}$. Then A' is open and closed and $g^+ \geqq \chi_{A'} \geqq 0$. Thus $e^+ \geqq i'\chi_{A'} \geqq 0$. However $f^- \geqq f^- - \chi_{A'} \geqq 0$ so that $e^- \geqq i'\chi_{A'} \geqq 0$. Thus $i'\chi_{A'}(k) = 0$ if $e^+(k) = 0$ or $e^-(k) = 0$, or $i'\chi_{A'} = 0$. It follows that $A' \subset G$ so that $A' = \emptyset$. Interchanging f and g in this argument yields $f = g$.

From these calculations one has that, given an open and closed set $A \subset K$, there is a unique open and closed set $A' \subset H - G$ such that $i'\chi_{A'} = \chi_A$ (let $e = \chi_A - \chi_{K-A}$ and select $\chi_{A'} = f^+$ as above). If A and B are open and closed and if $A \cap B = \emptyset$, it is easy to see that $(A \cup B)' = A' \cup B'$ and that $A' \cap B' = \emptyset$. If $A \cap B \neq \emptyset$ write $(A \cup B)'$

$=((A-B)\cup(A\cap B)\cup(B-A))'=(A-B)'\cup(A\cap B)'\cup(B-A)'$ $=[(A-B)'\cup(A\cap B)']\cup[(A\cap B)'\cup(B-A)']=A'\cup B'$. Noting that $K'=H-G$ and that $(K-A)'=H-(A'\cup G)$ one gets, by considering complements, that $(A\cap B)'=A'\cap B'$ for all open and closed sets A, $B\subset K$. Thus "$'$" preserves the ring operations of the ring of open and closed subsets of K.

Now let S be the submanifold of functions in $C(K)$ assuming only finitely many values. Define T: $S\rightarrow C(H)$ by $T(\sum_1^n a_i\chi_{A_i})=\sum_1^n a_i\chi_{A_i'}$ for $s=\sum_1^n a_i\chi_{A_i}\in S$. Since "$'$" preserves the ring operations it easily follows that T is linear, positive and that $\|Ts\|=\|s\|$ for all $s\in S$. Now S is dense in $C(K)$ as follows. If $\epsilon>0$ the set $\{k\mid -\|f\|+n\epsilon<f(k)<-\|f\|+(n+1)\epsilon\}$, $n=0, 1, 2, \cdots$, has open and closed closure A_n. The set B_n of k such that $f(k)=-\|f\|+n\epsilon$ and $k\notin A_{n-1}\cup A_n$ is also open and closed. At most a finite number of A_n, B_n are nonempty so $\|\sum_1^M(-\|f\|+n\epsilon)\chi_{A_n}+\sum_1^M(-\|f\|+n\epsilon)\chi_{B_n}-f\|\leqq\epsilon$ if M is large. Thus T has an extension to all of $C(K)$ (also denoted by T) which is positive and an isometry.

Since $(i'T\chi_A)(\nu)=T\chi_A(i\nu)=\chi_{A'}(i\nu)=i'\chi_{A'}(\nu)=\chi_A(\nu)$ for all ν in $N(K)$ and all open and closed sets $A\subset K$ one has that $i'T$ is the identity on $C(K)$. Q.E.D.

Let q be the natural embedding of E in E'' or of L in L''. Thus $q\nu(f)=f(\nu)$ for all f in E', ν in E (or f in L', ν in L).

THEOREM 3. *There is a norm one projection from E'' onto $q(E)$.*

Suppose for the moment this theorem has been proved. Let T be the isometry $E'\rightarrow L'$ promised in Theorem 2. The inclusion mapping i is suppressed in the following argument. Then T': $L''\rightarrow E''$ and for x in E one has that $T'qx=qx$ since $T'qx(x')=qx(Tx')=Tx'(x)=x'(x)$ $=qx(x')$ for every x' in E'. Thus $T'q(L)\supset q(E)$ in E''. By Theorem 3 there is a projection P of E'' onto $q(E)$ such that $\|P\|=1$. Then P restricted to $T'q(L)$ is a projection of $T'q(L)$ onto $q(E)$. Finally $Q=q^{-1}PT'q$ is a projection of L onto E having norm one since clearly $\|Q\|=1$, Q: $L\rightarrow E$, and Q is the identity on E. Thus one has

THEOREM 4. *If E is a closed sublattice of the (AL)-space L there is a projection Q of L onto E such that $\|Q\|=1$.*

PROOF OF THEOREM 3. Identify E'' with the space $R(K)$ so that $q(E)$ is identified with $N(K)$. It is sufficient to show there is a norm one projection of $R(K)$ onto $N(K)$. Let $\mathfrak{N}$ be the set of closed nowhere dense subsets of K. Let $\nu\geqq 0$ be in $R(K)$. Define ν_1 on an open and closed set A by $\nu_1(A)=\sup\{\nu(N)\mid N\subset A,\ N\in\mathfrak{N}\}$. Then ν_1 is finitely additive on the ring of open and closed sets. If $\sum_1^n a_i\chi_{A_i}\in S$, then

$\nu_1(\sum_1^n a_i\chi_{A_i}) = \sum_1^n a_i\nu(A_i)$ defines a continuous linear functional on S whose extension to $C(K)$ yields an element $\nu_2 \leqq \nu$ of $R(K)$. We will show that $\nu(N)=\nu_2(N)$ for all $N \in \mathfrak{N}$. Choose an open set $B \supset N$ such that $\nu_2(B-N) < \epsilon$. Choose f in $C(H)$ such that $\|f\|=1, f(k)=1$ if $k \in K-B$ and $f(k)=0$ if $k \in N$. $A = \mathrm{Cl}(\{k \mid f(k) < \frac{1}{2}\})$ is an open and closed set for which $\nu_2(A-N) < \epsilon$ and $N \subset A$. Then $\nu_2(N) \leqq \nu(N) \leqq \nu_2(A) \leqq \nu_2(N) + \epsilon$ so that $\nu_2(N) = \nu(N)$. Let $\nu_3 = \nu - \nu_2$. Clearly $0 \leqq \nu_3 \leqq \nu$ and $\nu_3 \in N(K)$. If we define $P(\nu) = \nu_3$ for $\nu \geqq 0$ then $P(a\nu) = aP(\nu)$ if $a \geqq 0$ and $P(\nu+\mu) = P(\nu) + P(\mu)$, $\nu,\mu \geqq 0$. Now any ν can be written $\nu = \mu - \lambda$ for some λ, $\mu \geqq 0$, and we define $P(\nu) = P(\mu) - P(\lambda)$. If $\nu = \mu_1 - \lambda_1 = \mu_2 - \lambda_2$ in this way then $\mu_1 + \lambda_2 = \mu_2 + \lambda_1$ so $P(\mu_1) + P(\lambda_2) = P(\mu_2) + P(\lambda_1)$ or $P(\mu_1) - P(\lambda_1) = P(\mu_2) - P(\lambda_2)$ and thus P is well defined. Moreover P is clearly linear and $\|P\| = 1$. Q.E.D.

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OHIO STATE UNIVERSITY