

## THE KERVAIRE INVARIANT OF $(8k+2)$ -MANIFOLDS

BY EDGAR H. BROWN, JR. AND FRANKLIN P. PETERSON<sup>1</sup>

Communicated by W. S. Massey, August 17, 1964

**1. Statements of results.** Let  $\Omega_m(\text{Spin})$ ,  $\Omega_m(SU)$ , and  $\Omega_m(e)$  denote the  $m$ th spinor, special unitary, and framed cobordism groups respectively (see [6]). In [4] Kervaire defined a homomorphism  $\Phi: \Omega_{2n}(e) \rightarrow Z_2$  for  $n$  odd and  $n \neq 1, 3$ , or  $7$ , and showed that  $\Phi=0$  for  $n=5$ . Kervaire and Milnor state in [5] that  $\Phi=0$  for  $n=9$ . One of the corollaries of our results is that  $\Phi=0$  for  $n=4k+1$ ,  $k \geq 1$ .

In [2] a homomorphism  $\Psi: \Phi_{2n}(\text{Spin}) \rightarrow Z_2$  was defined for  $2n = 8k+2$ ,  $k \geq 1$ , such that  $\Phi = \Psi\rho$ , where  $\rho: \Omega_{2n}(e) \rightarrow \Omega_{2n}(\text{Spin})$  is the obvious map.  $\Psi$  induces a map from  $\Omega_{2n}(SU)$  into  $Z_2$  which we also denote by  $\Psi$ . It is easily verified that  $\Omega_1(SU) = \Omega_1(\text{Spin}) = \Omega_1(e) = Z_2$ . Let  $\alpha$  be the generator. Let  $\theta$  be the secondary cohomology operation coming from the relation  $Sq^2Sq^2=0$  on an integer cohomology class [7]. If  $f$  is a map, let  $\theta_f$  denote the associated functional cohomology operation [7].

The main theorems of this announcement are the following.

**THEOREM 1.1.** *If  $\beta \in \Omega_{8k}(\text{Spin})$  and  $k \geq 1$ , then  $\Psi(\alpha^2 \cdot \beta) = \chi(\beta)$ , where  $\chi(\beta)$  is the Euler characteristic of  $\beta$  reduced mod 2.*

**THEOREM 1.2.** *If  $\beta \in \Omega_{8k}(SU)$  and  $k \geq 1$ , then  $\Psi(\alpha^2 \cdot \beta) = \theta_\nu(v^2)(M)$ , where  $M$  is a 3-connected manifold representing  $\alpha^2 \cdot \beta$ ,  $\nu: M \rightarrow BSU$  is the classifying map of the  $SU$ -structure on the normal bundle of  $M$ , and  $v \in H^{4k}(BSU; Z)$  is such that  $v$  reduced mod 2 is  $v_{4k}$  in the expression  $\overline{W} = Sq(1 + v_2 + v_4 + \dots)$ .*

We now deduce some corollaries of these two theorems.

**COROLLARY 1.3.**  $\Phi: \Omega_{8k+2}(e) \rightarrow Z_2$  is zero if  $k \geq 1$ .

**PROOF.** Conner and Floyd [9] and Lashof and Rothenberg<sup>2</sup> have shown that if  $\gamma \in \Omega_{8k+2}(SU)$  goes to zero in  $\Omega_{8k+2}(U)$ , then  $\gamma = \alpha^2 \cdot \beta$ , where  $\beta \in \Omega_{8k}(SU)$ . In particular, if  $\gamma = \bar{p}(\delta)$ ,  $\delta \in \Omega_{8k+2}(e)$ , then  $\gamma = \alpha^2 \cdot \beta$ . Let  $\delta = [M]$ . Then  $M$  can be taken to be 3-connected and  $\nu$  is homotopic to a constant. The corollary now follows from Theorem 1.2.

<sup>1</sup> The first named author was partially supported by the National Science Foundation and the second named author was partially supported by the U. S. Army Research Office and the National Science Foundation.

<sup>2</sup> We would like to thank R. Lashof and M. Rothenberg for communicating this result to us and for some helpful discussions.

**COROLLARY 1.4.**  *$bP_{8k+2}$ , the group of homotopy spheres bounding  $(8k+2)$ -dimensional  $\pi$ -manifolds, is isomorphic to  $Z_2$  if  $k \geq 1$ .*

**PROOF.** See [5, p. 536].

**COROLLARY 1.5.** *If  $M^{8k+2}$  is the closed combinatorial manifold obtained by plumbing together two copies of the tangent disc bundle of  $S^{4k+1}$  and attaching on  $(8k+2)$ -disc, then  $M^{8k+2}$  does not have the homotopy type of a differentiable manifold.*

**PROOF.** See [4].

**COROLLARY 1.6.** *Any element of  $\Omega_{8k+2}(e)$ ,  $k \geq 1$ , can be represented by a homotopy  $(8k+2)$ -sphere.*

**PROOF.** See [4].

The final corollary is in a different direction.

**COROLLARY 1.7.** *There exists  $\gamma \in \Omega_{8k+2}(\text{Spin})$  and  $\delta \in \Omega_{16}(SU)$  such that  $\Psi(\gamma) = \Psi(\delta) = 1$ . Furthermore,  $\gamma$  and  $\delta$  can be chosen to be orientably cobordant to zero and hence  $\Psi$  cannot be factored through  $\Omega_{8k+2} = \Omega_{8k+2}(SO)$ .*

**PROOF.** Let  $\gamma = \alpha^2 \cdot \beta$ , where  $\beta$  is a product of even-dimensional quaternionic projective spaces and use Theorem 1.1. By an analysis of the Adams spectral sequence for  $MSU$ , one may show that there is a  $\beta' \in \Omega_{16}(SU)$  with  $\chi(\beta') = 1$ . Let  $\delta = \alpha^2 \cdot \beta'$  and use Theorem 1.1.

**2. Outline of proofs.** Throughout this section all cohomology groups have  $Z_2$  coefficients and  $n = 4k + 1$ ,  $k \geq 1$ . We will only consider  $SU$ -cobordism because the cohomology operations are simpler in this case; the spinor case of Theorem 1.1 is a fairly easy generalization. All manifolds will be assumed to have an  $SU$ -structure on their normal bundle and if  $M$  is such a manifold, then  $\nu(M): M \rightarrow BSU(q)$ ,  $T(M): T(\nu) \rightarrow MSU(q)$ , and  $f(M): S^{m+2q} \rightarrow MSU(q)$  will denote, respectively, the classifying map, the associated map on the Thom spaces, and the Thom construction associated to the  $SU$ -structure.  $S^1$  will have the  $SU$ -structure so that  $[S^1] = \alpha \in \Omega_1(SU)$ .

In [3] it is shown how the relation  $Sq^2Sq^{n-1} + Sq^1(Sq^2Sq^{n-2}) = Sq^{n+1}$  leads to a quadratic cohomology operation

$$\begin{aligned} \phi: H^n(X) \cap \text{Ker } Sq^{n-1} \cap \text{Ker } Sq^2Sq^{n-2} \\ \rightarrow H^{2n}(X)/Sq^1(H^{2n-1}(X)) + Sq^2(H^{2n-2}(X)). \end{aligned}$$

If  $M$  is a closed  $2n$ -manifold with Stiefel-Whitney classes  $W_1(M) = 0$  and  $W_2(M) = 0$ , then  $\phi: H^n(M) \cap \text{Ker } Sq^{n-1} \rightarrow H^{2n}(M)$ . We define a function  $\lambda: \Omega_{2n}(K(Z_2, n); SU) \rightarrow Z_2$  as follows. If  $u \in H^n(M)$  and

$Sq^{n-1}(u) = 0$ , then  $\lambda(\{M, u\}) = \phi(u)(M)$ . If  $Sq^{n-1}(u) \neq 0$ , we change  $(M, u)$  in its bordism class so that  $Sq^{n-1}(u) = 0$ , e.g. make  $M$  simply connected. Then  $\Psi(M) = \sum \lambda(\{M, u_i\}) \cdot \lambda(\{M, v_i\})$ , where  $\{u_i, v_i\}$  is a symplectic basis for  $H^n(M)$  (see [2] and [3] for details of  $\phi$  and [5] for details of symplectic bases).

Let  $N$  be a 1-connected closed  $(2n-2)$ -manifold. Let  $P$  be 1-connected and  $M$  be 3-connected with  $[P] = \alpha \cdot [N]$  and  $[M] = \alpha^2 \cdot [N]$ . Let  $\eta$  be an appropriate suspension of the Hopf map from  $S^3$  to  $S^2$ . Note that  $f(M) \simeq f(N)\eta\eta$  and  $f(P) \simeq f(N)\eta$ . Let  $x \in H^1(S^1)$  be the generator, let  ${}^1\phi$  be the suspension of  $\phi$ , and let  $U_q$  be the Thom class of whatever Thom space is appropriate.

Theorem 1.2 is proved by verifying each of the following "equalities." In each case, the indeterminacy is zero and the element described lies in a group isomorphic to  $Z_2$ .

$$\begin{aligned} \Psi(N \times S^1 \times S^1) &= \lambda(N \times S^1 \times S^1, \nu(N)^*(v) \otimes 1 \otimes x) \\ &= \lambda(P \times S^1, \nu(P)^*(v) \otimes x) = \phi(\nu(P)^*(v) \otimes x) \\ &= {}^1\phi(\nu(P)^*(v)) = Sq_{\nu(P)}^2(v^2) = Sq_{\nu(P)}^2(v^2) \cdot U_q \\ &= Sq_{\tau(P)}^2(v^2 \cdot U_q) = Sq_{f(P)}^2(v^2 \cdot U_q) \\ &= Sq_{\eta}^2(f(N)^*(v^2 \cdot U_q)) = Sq_{\eta}^2 Sq_{\eta}^2(f(N)^*(v^2 \cdot U_q)) \\ &= \theta_{\eta\eta}(f(N)^*(v^2 \cdot U_q)) = \theta_{f(M)}(v^2 \cdot U_q) \\ &= \theta_{\nu(M)}(v^2) \cdot U_q = \theta_{\nu(M)}(v^2). \end{aligned}$$

The first equality is obtained from a careful choice of a symplectic basis for  $N \times S^1 \times S^1$ . The fifth, seventh, eleventh, and fourteenth equalities follow respectively from results in [8], [1], [7], and [1]. Theorem 1.1 follows from the equation  $\Psi(N \times S^1 \times S^1) = Sq_{\eta}^2(f(N)^*(v^2 \cdot U_q))$  and the facts that  $Sq_{\eta}^2$  is an isomorphism and  $\nu(N)^*(v^2) = \chi(N)$ .

### BIBLIOGRAPHY

1. J. Adem, *Sobre operaciones cohomológicas secundarias*, Bol. Soc. Mat. Mexicana 7 (1962), 71-84.
2. E. H. Brown, Jr., *Note on an invariant of Kervaire*, Michigan Math. J. (to appear).
3. E. H. Brown, Jr. and F. P. Peterson, *Whitehead products and cohomology operations*, Quart. J. Math. Oxford Ser. (2) 15 (1964), 116-120.
4. M. Kervaire, *A manifold which does not admit any differentiable structure*, Comment. Math. Helv. 34 (1960), 257-270.
5. M. Kervaire and J. W. Milnor, *Groups of homotopy spheres. I*, Ann. of Math. (2) 77 (1963), 504-537.

6. J. Milnor, *Spin structures on manifolds*, Enseignement Math. **9** (1963), 198–203.
7. F. P. Peterson, *Functional cohomology operations*, Trans. Amer. Math. Soc. **86** (1957), 197–211.
8. F. P. Peterson and N. Stein, *Secondary cohomology operations: two formulas*, Amer. J. Math. **81** (1959), 281–305.
9. P. E. Conner and E. E. Floyd, *The SU-bordism theory*, Bull. Amer. Math. Soc. **70** (1964), 670–675.

BRANDEIS UNIVERSITY AND  
MASSACHUSETTS INSTITUTE OF TECHNOLOGY

#### STATEMENT OF OWNERSHIP, MANAGEMENT AND CIRCULATION

Act of October 23, 1962: Section 4369, Title 39, United States Code

1. *Date of Filing*: September 22, 1964
2. *Title of Publication*: Bulletin of the American Mathematical Society
3. *Frequency of issue*. Six times per year
4. *Location of Known Office of Publication*: 190 Hope Street, Providence, Rhode Island 02906
5. *Location of the Headquarters or General Business Offices of the Publishers*: Same
6. *Names and Addresses of Publisher, Editor, and Managing Editor*. *Publisher*: American Mathematical Society, 190 Hope Street, Providence, Rhode Island 02906. *Editor*: Felix Browder, Chairman of the Editorial Committee, 190 Hope Street, Providence, Rhode Island 02906. *Managing Editor*: None
7. *Owner*: None
8. *Known Bondholders, Mortgagees and Other Security Holders Owning or Holding 1 Percent or More of Total Amount of Bonds, Mortgages or Other Securities*: None
9. Paragraphs 7 and 8 include, in cases where the stockholder or security holder appears upon the books of the company as trustee or in any other fiduciary relation, the name of the person or corporation for whom such trustee is acting, also the statements in the two paragraphs show the affiant's full knowledge and belief as to the circumstances and conditions under which stockholders and security holders who do not appear upon the books of the company as trustees, hold stock and securities in a capacity other than that of a bona fide owner. Names and addresses of individual who are stockholders of a corporation which itself is a stockholder or holder of bonds, mortgages or other securities of the publishing corporation have been included in paragraphs 7 and 8 when the interests of such individuals are equivalent to 1 percent or more of the total amount of the stock or securities of the publishing corporation.

I certify that the statements made by me above are correct and complete.—Gordon L. Walker