STIELTJES INTEGRATION, SPECTRAL ANALYSIS, AND THE LOCALLY-CONVEX ALGEBRA (BV)

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The space (BV) (of all functions of bounded variation on an interval $[b_0, b_1]$) is an algebra under pointwise multiplication; one of our aims is to show how it can be made into a locally-convex algebra on which all continuous linear multiplicative functionals are represented by point measures on the interval $(b_0, b_1]$. Our main result deals with spectral and non spectral operators.

The algebra structure is disregarded in §5, where (BV) is endowed with a topology such that the most general continuous linear functional on (BV) has a natural representation by means of a Stieltjes integral.

Given a complete barreled space $\mathfrak{X}$, we introduce a family $\mathcal{E}_1$ of functions whose values are commuting projection operators in $\mathfrak{X}$. The algebra (BV) is topologized in such a way that each strongly-continuous representation $g \rightarrow u(g)$ (of (BV) on $\mathfrak{X}$) can be expressed in a natural way in terms of some $F \in \mathcal{E}_1$; in fact, $u(g)$ is the Stieltjes integral of g with respect to F.

1. A Helly theorem for Stieltjes integrals. Let $\mathfrak{A}$ be an arbitrary complete locally-convex Hausdorff linear space; let F be a bounded function on the interval $[b_0, b_1]$ into $\mathfrak{A}$, and let g belong to the space (BV) of all complex-valued functions of bounded variation on $[b_0, b_1]$. Our basic theorem is as follows: if

(i) *the left-hand limit* $F(\alpha - 0)$ *exists whenever* $\alpha > b_0$,

(ii) $F(\beta)$ = *the right-hand limit* $F(\beta + 0)$ *whenever* $\beta < b_1$,

(iii) $F(b_1) = F(b_1 - 0)$ *and* $F(b_0)$ = *the zero-element of* $\mathfrak{A}$,

then the Stieltjes sums

$$\sum_{k=1}^{n} g(x_k)\{F(x_k) - F(x_{k-1})\}$$

(where $-\infty \leqq b_0 = x_0 < x_1 < \cdots < x_n = b_1 \leqq \infty$) converge to a limit, here denoted

$$\int g(\oplus[\lambda]) \cdot dF(\lambda) \tag{1}$$

—the limit is to be understood in the sense of refinements of subdivisions of $[b_0, b_1]$. If g is left-continuous, then (1) coincides with the usual Stieltjes integral. The integral (1) satisfies an integration-by-parts formula involving a modified σ-Stieltjes integral (in the sense of Hildebrandt [1, p. 273]). The existence of the integral (1) implies the following Helly theorem:[1]

Let $(g_z: z)$ *be a net in* (BV) *such that the set of total variations is bounded. If* $g\in$(BV) *is such that* $g(\lambda)=\lim_z g_z(\lambda)$ *whenever* $b_0<\lambda\leqq b_1$, *then*

$$(2)\qquad \int g(\oplus[\lambda])\cdot dF(\lambda) = \lim_z \int g_z(\oplus[\lambda])\cdot dF(\lambda) \quad (\textit{convergence in } \mathfrak{A}).$$

2. **The spectral theorem.** Let $\mathfrak{L}(\mathfrak{X}, \mathfrak{X})$ be the algebra of all linear continuous operators in some complete barreled Hausdorff linear space $\mathfrak{X}$. Henceforth, $\mathfrak{A}$ will be either a Banach algebra or the algebra $\mathfrak{L}(\mathfrak{X}, \mathfrak{X})$ endowed with the strong operator-topology; although $\mathfrak{A}$ need not be topologically complete, we have the

THEOREM I. *All the results of* §1 *are still valid.*

Let $\mathcal{E}_0$ be the family of all bounded, $\mathfrak{A}$-valued functions F on $[b_0, b_1]$ which satisfy all three conditions (i)–(iii), and let $\mathcal{E}_1$ be the family of all $F\in\mathcal{E}_0$ such that $F(b_1)=I$ (the identity-operator), and

$$F(\alpha)F(\beta) = F(\alpha) = F(\beta)F(\alpha), \quad \text{whenever } \alpha \leqq \beta.$$

The above juxtaposition $F(\alpha)F(\beta)$ indicates either multiplication in the algebra $\mathfrak{A}$, or the usual composition of operators–when $\mathfrak{A}=\mathfrak{L}(\mathfrak{X}, \mathfrak{X})$.

It will be convenient to say that a net $(g_z: z)$ *converges* $\mathcal{E}_1$-*weakly* to a function $g\in$(BV) if (and only if) relation (2) holds for any $F\in\mathcal{E}_1$.

If $H\in\mathfrak{A}$ and if g belongs to the family P of all polynomials, we write

$$g(H) = g(0)I + g'(0)H + \frac{1}{2!}g^{(2)}(0)H^2 + \frac{1}{3!}g^{(3)}(0)H^3 + \cdots.$$

Consider the transformation $g\to g(H)$ of P into $\mathfrak{A}$; by definition, it is $\mathcal{E}_1$-*continuous* if (and only if), for any $g\in P$, the relation

$$\lim_z g_z(H) = g(H) \qquad (\text{convergence in } \mathfrak{A})$$

obtains whenever $(g_z: z)$ is a net in P which converges $\mathcal{E}_1$-weakly to g. When H is a spectral operator (in the sense of Schaefer [4, p. 155]),

[1] I am indebted to T. H. Hildebrandt who sent me, in addition to detailed information, his own proof relating to the case where F is scalar-valued.

then the transformation $g \to g(H)$ is $\mathcal{E}_1$-continuous; this happens, in particular, when H is a self-adjoint operator.

THEOREM II. *Let $H \in \mathfrak{A}$ be such that the transformation $g \to g(H)$ is $\mathcal{E}_1$-continuous. There exists an $\mathcal{E}_1$-continuous linear transformation (of* (BV) *into $\mathfrak{A}$) which maps the polynomial $p(\lambda) = \lambda$ onto the operator H; this transformation, denoted $g \to u(g)$, is the unique $\mathcal{E}_1$-continuous extension to all of* (BV) *of the transformation $g \to g(H)$. Further, the transformation $g \to u(g)$ is an algebra-homomorphism of the algebra* (BV) *(under pointwise multiplication), and*

$$u(g) = \int g(\oplus[\lambda]) \cdot dF(\lambda) \qquad (\textit{for all } g \in (\mathrm{BV})),$$

where F is the unique element of $\mathcal{E}_1$ such that $H = \int \lambda \cdot dF(\lambda)$. If $\lambda \in [b_0, b_1]$ and $g \in$ (BV), then $F(\lambda)$ and $u(g)$ belong to the family of all the elements Q of $\mathfrak{A}$ such that

$$TQ = QT \quad \textit{whenever} \quad TH = HT \quad \textit{and} \quad T \in \mathfrak{A}.$$

Finally, if g is continuous and $g \in$ (BV), then the spectrum of $u(g)$ is the image $g(\sigma(H))$ of the spectrum $\sigma(H)$.

The hypothesis of Theorem II is satisfied when H is the Hilbert operator H_p in $\mathfrak{X} = L^p(-\infty, \infty)$, or its analogue in the sequence space l_p, where $1 < p < \infty$ and $\mathfrak{A}$ is the algebra $\mathfrak{L}(\mathfrak{X}, \mathfrak{X})$ of bounded linear operators in $\mathfrak{X}$ (both operators are essentially convolution with the function $f(\lambda) = 1/\lambda$, $f(0) = 0$). When $p \neq 2$, both operators H_p fall outside existing theories: for example, they are not spectral operators, although they are self-adjoint when $p = 2$.

In case $\mathfrak{X}$ is a reflexive Banach space, the conclusions of Theorem II are stronger than the conclusions obtained from the weaker assumption that H is a "well-bounded operator" (in the sense of Smart [3], [5]).

3. Topological considerations. We shall now attempt to endow (BV) with a topology such that $\mathcal{E}_1$-convergence coincides with convergence in the sense of that topology. In order that this topology be Hausdorff, it is necessary to identify functions whose values may differ outside the half-open interval $(b_0, b_1]$. Consequently, (BV) will henceforth be replaced by the family $V^b = \{g^b : g \in (\mathrm{BV})\}$, where g^b denotes the restriction of the function g to the half-open interval $(b_0, b_1]$. Accordingly, $G \in V^b$ implies that $G = g^b$ for some $g \in$ (BV) (that is, $G(\lambda) = g(\lambda)$ for $b_0 < \lambda \leq b_1$); the integral (1) depends on G

and not on g, so that we may introduce the following abbreviation:

$$u^F(G) = \int g(\oplus[\lambda]) \cdot dF(\lambda). \tag{3}$$

Let $\{\|\cdot\|_i : i \in I\}$ be a family of semi-norms determining the topology of $\mathfrak{A}$, and let V_1^b be the space V^b endowed with the topology determined by the semi-norms $G \to \|u^F(G)\|_i$, where $i \in I$ and $F \in \mathcal{E}_1$. It is not hard to verify that V_1^b is a locally-convex Hausdorff linear space, and since $\mathcal{E}_1$-convergence coincides with the notion of convergence in the topology of V_1^b, a transformation on V_1^b is continuous if (and only if) it is $\mathcal{E}_1$-continuous. It might be noted that polynomials are dense in V_1^b. See §5 for another construction of V_1^b: it suffices to replace $\mathcal{E}_0$ by $\mathcal{E}_1$ in §5 to obtain V_1^b.

4. **Continuous algebra-homomorphisms.** As in Theorem II, we exploit the fact that V^b is an algebra under pointwise multiplication ($G_1G_2(\lambda) = G_1(\lambda)G_2(\lambda)$). Let $\mathrm{Hom}(V_1^b : \mathfrak{A})$ denote the family of all continuous algebra-homomorphisms of V_1^b; thus, $u \in \mathrm{Hom}(V_1^b : \mathfrak{A})$ if (and only if) u is a continuous linear transformation of the topological linear space V_1^b into $\mathfrak{A}$, such that $u(G_1G_2) = u(G_1)u(G_2)$ when $G_1, G_2 \in V_1^b$ and $u(1) =$ the identity-operator.

Let u^F denote the transformation $G \to u^F(G)$ defined by equation (3); it is a continuous algebra-homomorphism of V_1^b; in fact, the following theorem shows it to be the most general element of the family $\mathrm{Hom}(V_1^b : \mathfrak{A})$.

THEOREM III. *If u is a continuous algebra-homomorphism of V_1^b, there exists one and only one function $F \in \mathcal{E}_1$ such that $u(G) = u^F(G)$ for all G in V^b. The mapping $F \to u^F$ is a one-to-one correspondence of $\mathcal{E}_1$ onto* $\mathrm{Hom}(V_1^b : \mathfrak{A})$.

As will be seen in §6, the topology of V_1^b is boundedly compatible with the algebra V^b. In case $\mathfrak{A}$ is the complex field **C**, then

$$\mathrm{Hom}(V_1^b : \mathbf{C}) = \{u^\alpha : b_0 < \alpha \leqq b_1\},$$

where u^α is defined on V^b by the relation $u^\alpha(G) = G(\alpha)$, $G \in V^b$.

5. **Integral representation of continuous linear transformations.** As in §2, let $\mathcal{E}_0$ be the family of all bounded, $\mathfrak{A}$-valued functions F on $[b_0, b_1]$ which satisfy all three conditions (i)–(iii); we shall now use $\mathcal{E}_0$ to define on V^b a topology that is finer than the topology of V_1^b. Set $G \in V^b$ and consider the mapping $F \to u^F(G)$ (of $\mathcal{E}_0$ into $\mathfrak{A}$) defined

by equation (3); it is an element (denoted G^*) of the family $L(\mathcal{E}_0, \mathfrak{A})$ of all linear mappings of $\mathcal{E}_0$ into $\mathfrak{A}$. Let $L(\mathcal{E}_0, \mathfrak{A})$ be endowed with the topology of simple convergence on $\mathcal{E}_0$; the transformation $G \rightarrow G^*$ identifies V^b with a subset of $L(\mathcal{E}_0, \mathfrak{A})$, and V_0^b is defined as the space V^b endowed with the topology induced on it by the topology of $L(\mathcal{E}_0, \mathfrak{A})$.

Set $F \in \mathcal{E}_0$; the transformation $G \rightarrow u^F(G)$ (defined by equation (3)) is a continuous linear transformation of V_0^b into $\mathfrak{A}$. In fact, we have

THEOREM IV.[2] *If u is a continuous linear transformation of V_0^b into $\mathfrak{A}$, there exists one and only one function $F \in \mathcal{E}_0$ such that*

$$u(g^b) = \int g(\oplus[\lambda]) \cdot dF(\lambda) \qquad (\textit{for all } g \in (\mathrm{BV})).$$

We recall that g^b is the restriction of g to the half-open interval. Let $\mathfrak{A}$ now be the complex field and $-\infty < b_0 < b_1 < \infty$: the transformation $F \rightarrow u^F$ (see Theorem III) identifies $\mathcal{E}_0$ with the dual $(V_0^b)^*$ of V_0^b, whence $\mathcal{E}_0$ can be identified with a subset of the bidual C^{**} of the Banach space C of continuous functions on $[b_0, b_1]$.

6. **Locally-convex algebras.** Let $\mathfrak{R}$ be an algebra endowed with a topology such that, for any two bounded and converging nets in $\mathfrak{R}$, the product of the limits is the limit of the product. It will be convenient to describe this situation by saying that *the topology of $\mathfrak{R}$ is boundedly compatible with the algebra $\mathfrak{R}$*. Note that such an $\mathfrak{R}$ is a "locally-convex algebra" in the sense of Schaefer [4].

For example, the norm-topology is boundedly compatible with the algebra V^b (under pointwise multiplication). Again, let V_1^b be the locally-convex algebra obtained in §3; we have the

THEOREM V. *The topology of V_1^b is boundedly compatible with the algebra V^b.*

THEOREM VI. *Let $\mathfrak{A}$ be an arbitrary algebra with unit; Theorems* II–III *are valid whenever $\mathfrak{A}$ is endowed with a locally-convex Hausdorff linear topology which is boundedly compatible with the algebra $\mathfrak{A}$.*

Let $\mathfrak{A}$ be the algebra $\mathfrak{L}(\mathfrak{X}, \mathfrak{X})$ that was defined in §2; it is not hard to see that the strong operator-topology is boundedly compatible with the algebra $\mathfrak{A}$.

Added in proof. The verification of Theorem I is contained in a

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[2] This theorem cannot be inferred from the standard duality theorem; Frank Ryan constructed the counterexample.

paper entitled *A Helly convergence theorem for Stieltjes integrals* (by Krabbe) which will appear in Nederl. Akad. Wetensch. Proc. Ser. A, communicated by Professor J. Ridder on 25 September 1964.

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