WHITEHEAD GROUPS OF FREE ASSOCIATIVE ALGEBRAS

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Let R be a principal ideal domain, X a set, and Λ the free associative algebra over R on the set X. Then Λ is a supplemented algebra over R, where the augmentation $\epsilon_\Lambda: \Lambda \to R$ is the unique map of algebras extending $x \to 0$, $x \in X$, given by the universal property of Λ. We denote $\overline{K}_1(\Lambda) = \text{coker } \eta_{\Lambda*}: K_1(R) \to K_1(\Lambda)$, where $\eta: R \to \Lambda$ is the unit.[1]

THEOREM 1. $\overline{K}_1(\Lambda) = 0$, *or, equivalently,* $\eta_{\Lambda*}: K_1(R) \to K_1(\Lambda)$ *is an isomorphism.*

We remark that Theorem 1 applies to the case $R = Z$, the ring of integers, or $R =$ any field. Since $\eta_{\Lambda*}$ is a monomorphism for functorial reasons ($\epsilon_\Lambda \eta_\Lambda = 1: R \to R$), the two assertions of Theorem 1 are seen to be equivalent.

LEMMA 1. *Any regular matrix T over Λ is equivalent by elementary operations to a regular matrix of the form*

$$M = M_0 + M_1 x_1 + M_2 x_2 + \cdots + M_n x_n,$$

where M_i $(0 \leqq i \leqq n)$ are matrices over R and $x_1, x_2, \cdots, x_n$ are distinct elements of X.

The proof is a standard exercise and will be omitted (see also [3]).

Using the notation of Lemma 1, if we apply ϵ_Λ, we see that M_0 is a regular matrix over R. Thus, $[M] = [M_0^{-1}M] \in \overline{K}_1(\Lambda)$, and $[M] \in \overline{K}_1(\Lambda)$ is represented by an $m \times m$ matrix of the form

$$N = 1 + N_1 x_1 + N_2 x_2 + \cdots + N_n x_n, \tag{1}$$

where N_i $(1 \leqq i \leqq n)$ are matrices over R, and $x_1, x_2, \cdots, x_n$ are distinct elements of X.

LEMMA 2. *The subalgebra (without unit) $\mathfrak{N}$, generated by $N_1, N_2, \cdots, N_n$, of the ring of endomorphisms $E(R, m)$ of a free R-module of rank m, is nilpotent.*

PROOF. Since N is regular, there is a matrix

[1] If R is a ring (associative, with unit), then $K_1(R) = \text{GL}(R)/\mathcal{E}(R)$ where $\text{GL}(R) = \text{dir}_\to\text{limit GL}(n, R)$ and $\mathcal{E}(R) = \text{dir}_\to\text{limit } \mathcal{E}(n, R)$, where $\mathcal{E}(n, R)$ is the subgroup of $\text{GL}(n, R)$ generated by elementary matrices (see Bass [1]).

$$A = A_0 + A_1x_1 + \cdots + A_nx_n + \sum_{i,j=1}^{n} A_{ij}x_ix_j$$
$$+ \sum_{i,j,k=1}^{n} A_{ijk}x_ix_jx_k + \cdots$$

over Λ, where the A_i, A_{ij}, $\cdots$, are matrices over R, such that $AN=1$. By equating coefficients of monomials in the x's, we derive the relations

$$\begin{aligned}
&A_0 = 1,\\
&A_i + N_i = 0 \Rightarrow A_i = -N_i, \qquad 1 \leqq i \leqq n,\\
&A_{ij} + A_iN_j = 0 \Rightarrow A_{ij} = N_iN_j, \qquad 1 \leqq i, j \leqq n,\\
&\quad\vdots\\
&A_{i_1i_2\cdots i_r} + A_{i_1}N_{i_2\cdots i_r} = 0 \Rightarrow A_{i_1i_2\cdots i_r} = (-1)^rN_{i_1}N_{i_2}\cdots N_{i_r},\\
&\qquad\qquad 1 \leqq i_i, i_2, \cdots, i_r \leqq n.
\end{aligned}$$

Since A is a finite sum of the terms indicated, we deduce that there is an r such that $N_{i_1}N_{i_2}\cdots N_{i_s}=0$, all $s \geqq r$, $1 \leqq i_1, i_2, \cdots, i_s \leqq n$. This fact, and the commutativity of R, establish that $\mathfrak{N}$ is nilpotent.

THEOREM 2. *If R is a principal ideal domain, any nilpotent subalgebra $\mathfrak{N}$ of $E(R, m)$ can be put in upper niltriangular form; that is, a basis for the free module R^m of rank m over R can be chosen so that any $T \in \mathfrak{N}$ is represented by a matrix $\{T_{ij}\}$ with $T_{ij}=0$ if $i \geqq j$.*

Assuming Theorem 2, let us prove Theorem 1. From Lemma 2 and Theorem 2, there is a regular matrix B over R such that $B^{-1}N_iB$ is upper niltriangular, $1 \leqq i \leqq n$. Thus

$$(2) \quad B^{-1}NB = 1 + B^{-1}N_1Bx_1 + B^{-1}N_2Bx_2 + \cdots + B^{-1}N_nBx_n,$$

and it is easy to reduce the matrix on the right-hand side of (2) to the identity by elementary column operations. Thus $[N] = [B^{-1}NB] = 0$ in $\overline{K}_1(\Lambda)$ and, since we started with an arbitrary regular T, and $[T] = [N]$, we have shown $[T] = 0$ in $\overline{K}_1(\Lambda)$.

PROOF OF THEOREM 2. The proof proceeds by induction on m. If $m=1$, the theorem is trivial, since R is a domain. Assume the theorem is true for $m=1, 2, \cdots, k-1$. We shall show it is true for $m=k$. Let V be a free R-module of rank k, $\mathfrak{N}$ a nilpotent subalgebra of $E(R, k)$ (we have chosen some basis of V so that $E(R, k)$ acts on V on the right). Then

$$V \supset V\mathfrak{N} \supset \cdots \supset V\mathfrak{N}^{r-1} \supset V\mathfrak{N}^r = 0$$

for some integer r. Now $V\mathfrak{N}$ is a submodule of a free R-module V of finite rank k, hence is free of rank $\leqq k$. In fact, by passing to the quotient field of R, and using the fact that $\mathfrak{N}^r=0$, we see that rank $V\mathfrak{N}$ is $<k$. Now by the theorem of elementary divisors (Bourbaki [2]), we can find bases

$$v_1, v_2, \cdots, v_i, v'_{i+1}, v'_{i+2}, \cdots, v'_k \text{ for } V,$$

$$u_{i+1}, u_{i+2}, \cdots, u_k \text{ for } V\mathfrak{N},$$

where $i\geqq 1$, and elements $r_{i+1}, r_{i+2}, \cdots, r_k\in R$ such that

$$u_{i+1} = r_{i+1}v'_{i+1}, u_{i+2} = r_{i+2}v'_{i+2}, \cdots, u_k = r_kv'_k.$$

Let V_1 be the submodule of V generated by $(v'_{i+1}, v'_{i+2}, \cdots, v'_k)$ so rank $V_1<k$. Now $V_1\mathfrak{N}\subset V\mathfrak{N}\subset V_1$, so $\mathfrak{N}$ can be considered a nilpotent algebra of endomorphisms acting on V_1. By the inductive hypothesis there is a basis $v_{i+1}, v_{i+2}, \cdots, v_k$ for V_1 so that each matrix of $\mathfrak{N}$ restricted to V_1 is in upper niltriangular form with respect to this basis. Extend $v_{i+1}, \cdots, v_k$ to the basis $v_1, \cdots, v_i$, $v_{i+1}, \cdots, v_k$ of V. Then it is easily seen that each matrix of $\mathfrak{N}$ is in upper niltriangular form with respect to this basis of V. This completes the proof.

The author does not know whether theorems more general than Theorems 1 and 2 are valid. To show that some restrictions on R are needed, let R be a commutative ring with a nilpotent element $a\in R$ (e.g., $R=\boldsymbol{Z}_4$, the ring of integers mod 4, $a=2$). Let $X=\{x\}$. Then the 1×1 matrix $1+ax$ is regular, but it is easily seen, by taking determinants, not to be in the image of $\eta_*: K_1(R)\to K_1(R[x])$. Furthermore the ideal aR is a nilpotent subalgebra of 1×1 matrices which cannot be put in niltriangular form.

References

1. H. Bass, *The stable structure of quite general linear groups*, Bull. Amer. Math. Soc. **70** (1964), 429–433.

2. N. Bourbaki, *Éléments de mathématique*. XIV, Livre II, Chap. VII, §4, 2, Theorem 1, Actualités Sci. Indust., Hermann, Paris, 1952.

3. G. Higman, *The units of group rings*, Proc. London Math. Soc. (2) **46** (1940), 231–248.

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