A RADON-NIKODYM THEOREM IN W^*-ALGEBRAS[1]

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1. **Introduction.** The purpose of this paper is to show a Radon-Nikodym theorem in general W^*-algebras as follows: Let M be a W^*-algebra, and ϕ, ψ two normal positive linear functionals on M such that $\psi \leqq \phi$; then there is a positive element t_0 of M with $0 \leqq t_0 \leqq 1$ satisfying $\psi(x) = \phi(t_0 x t_0)$ for all $x \in M$ (Theorem 2). This theorem is the affirmative solution to a problem raised by Dixmier [**1**, p. 63] and the author [**3**, p. 1.46 and Question 2 in the appendix]. A less cogent Radon-Nikodym theorem in general W^*-algebras has been proved by the author [**3**, p. 1.46].

2. **Theorems.** To prove the above theorem, we shall provide some considerations.

Let M be a W^*-algebra, ϕ a normal positive linear functional on M. For $a, x \in M$, put $(Ra\phi)(x) = \phi(xa)$; then $Ra\phi$ is a σ-continuous linear functional on M. Then we shall show

PROPOSITION 1. *Suppose that $Ra\phi$ is self-adjoint; then we have* $|(Ra\phi)(h)| = |\phi(ha)| \leqq \|a\|\phi(h)$ for $h\ (\geqq 0) \in M$.

PROOF. By the assumption, $(Ra\phi)^*(x) = [(Ra\phi)(x^*)]^- = [\phi(x^*a)]^- = [\phi((a^*x)^*)]^- = \phi(a^*x) = (Ra\phi)(x) = \phi(xa)$ for $x \in M$.

Hence $\phi(a^*x) = \phi(xa)$, so that $\phi(xa^2) = \phi(xaa) = \phi(a^*xa)$; therefore $Ra^2\phi \geqq 0$ and so, analogously, we have $\phi(xa^4) = \phi((a^2)^*xa^2)$.

By the analogous discussion, we have

$$\phi(xa^{2^{n+1}}) = \phi((a^{2^n})^* x (a^{2^n})) \quad \text{for } x \in M.$$

Then, for $h \geqq 0$,

$$\begin{aligned} |\phi(ha)| &= |\phi(h^{1/2}h^{1/2}a)| \leqq \phi(h)^{1/2}\phi(a^*ha)^{1/2} \\ &= \phi(h)^{1/2}\phi(ha^2)^{1/2} \leqq \phi(h)^{1/2}\{\phi(h)^{1/2}\phi((a^2)^*ha^2)^{1/2}\}^{1/2} \\ &= \phi(h)^{1/2}\phi(h)^{1/4}\phi(ha^4)^{1/4} = \phi(H)^{1/2+1/4}\phi(ha^4)^{1/4} \\ &= \cdots\cdots\cdots \\ &= \phi(h)^{\sum_{i=1}^{n}(1/2^i)}\,\phi(ha^{2^n})^{1/2^n} = \phi(h)^{1-1/2^n}\,\phi(ha^{2^n})^{1/2^n} \\ &\leqq \phi(h)^{1-1/2^n}(\|\phi\|\|h\|\|a\|^{2^n})^{1/2^n} \\ &= \phi(h)^{1-1/2^n}\|a\|(\|\phi\|\|h\|)^{1/2^n} \to \|a\|\phi(h) \qquad (n \to \infty). \end{aligned}$$

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Hence we have $|\phi(ha)| \leqq \|a\|\phi(h)$.

This completes the proof.

Now we shall show an application of Proposition 1. For $b \in M$, we consider a linear functional $Rb\phi$, then

THEOREM 1. *Let $Rb\phi = Rv|Rb\phi|$ be the polar decomposition of $Rb\phi$ (cf. [2], [3]); then the absolute value $|Rb\phi|$ of $Rb\phi$ is majorized by $\|b\|\phi$, that is, $|Rb\phi| \leqq \|b\|\phi$.*

PROOF. Since $|Rb\phi| = Rv^*(Rb\phi)$ (cf. [2], [3]), $|Rb\phi|(x) = \phi(xv^*b)$, so that by Proposition 1 we have

$$\begin{aligned} |\phi(hv^*b)| &= \phi(hv^*b) \leqq \|v^*b\|\phi(h) \\ &\leqq \|b\|\phi(h) \quad \text{for } h\ (\geqq 0) \in M. \end{aligned}$$

This completes the proof.

Now let $s(\phi)$ be the support of ϕ and we shall consider the W^*-algebra $s(\phi)Ms(\phi)$. Let $\tilde{\phi}$ be the restriction of ϕ on $s(\phi)Ms(\phi)$.

Let $\pi^{\tilde{\phi}}(\mathfrak{H}\tilde{\phi})$ be the W^*-representation of $s(\phi)Ms(\phi)$ on a Hilbert space $\mathfrak{H}\tilde{\phi}$ constructed via $\tilde{\phi}$, then we can consider $s(\phi)Ms(\phi)$ as a concrete W^*-algebra on the Hilbert space $\mathfrak{H}\tilde{\phi}$. Let ξ be the image of $s(\phi)$ in $\mathfrak{H}\tilde{\phi}$, then $\tilde{\phi}(x) = \langle x\xi, \xi\rangle$ for $x \in s(\phi)Ms(\phi)$, where $\langle\ ,\ \rangle$ is the inner product of $\mathfrak{H}\tilde{\phi}$.

Let $\{s(\phi)Ms(\phi)\}'$ be the commutant of $s(\phi)Ms(\phi)$ in $\mathfrak{H}\tilde{\phi}$, then $[s(\phi)Ms(\phi)\xi] = [\{s(\phi)Ms(\phi)\}'\xi] = \mathfrak{H}\tilde{\phi}$, where $[(\cdot)]$ is the closed linear subspace of $\mathfrak{H}\tilde{\phi}$ generated by $(\cdot)$, namely, ξ is a separating and generating vector of $s(\phi)Ms(\phi)$.

Now we shall show

THEOREM 2. *Let ψ be a normal positive linear functional on M such that $\psi \leqq \phi$; then there is a positive element t_0 of M with $0 \leqq t_0 \leqq 1$ satisfying $\psi(x) = \phi(t_0xt_0)$ for $x \in M$.*

PROOF. Let $\tilde{\psi}$ be the restriction of ψ on $s(\phi)Ms(\phi)$; then $\tilde{\psi} \leqq \tilde{\phi}$ and, therefore, there is a positive element h_0' with $\|h_0'\| \leqq 1$ of $\{s(\phi)Ms(\phi)\}'$ such that $\tilde{\psi}(x) = \langle xh_0'\xi, h_0'\xi\rangle$ for $x \in s(\phi)Ms(\phi)$.

Now we shall consider a σ-continuous linear functional f' on the W^*-algebra $\{s(\phi)Ms(\phi)\}'$ as follows: $f'(y') = \langle y'h_0'\xi, \xi\rangle$ for $y' \in \{s(\phi)Ms(\phi)\}'$; then $f' = R_{h_0'}g'$, where $g'(y') = \langle y'\xi, \xi\rangle$ for $y' \in \{s(\phi)Ms(\phi)\}'$.

Since $g' \geqq 0$, by Theorem 1, $|f'| \leqq \|h_0'\|g'$, so that there is a positive element t_0 of $s(\phi)Ms(\phi)$ with $0 \leqq t_0 \leqq 1$ such that $|f'|(y') = \langle y't_0\xi, \xi\rangle$.

Then

$$|f'|(y') = R_{(v')^*}f'(y') = f'(y'v'^*) = g'(y'v'^*h_0'),$$

where $R_{v'}|f'| = f'$ is the polar decomposition of f'.

Hence

$$\langle y' t_0 \xi, \xi \rangle = \langle y' v'^* h_0' \xi, \xi \rangle$$

for $y' \in \{s(\phi) M s(\phi)\}'$.

Since $[\{s(\phi) M s(\phi)\}' \xi] = \mathfrak{H}_{\tilde{\phi}}$, we have $t_0 \xi = v'^* h_0' \xi$ and so $v' t_0 \xi = v' v'^* h_0' \xi$.

On the other hand,

$$\begin{aligned}\langle y' v' v'^* h_0' \xi, \xi \rangle &= |f'|(y'v') = Rv'|f'|(y') = f(y') \\ &= \langle y' h_0' \xi, \xi \rangle \quad \text{for } y' \in \{s(\phi) M s(\phi)\}';\end{aligned}$$

hence $v' v'^* h_0' \xi = h_0' \xi$ and so $v' t_0 \xi = h_0' \xi$. Therefore,

$$\begin{aligned}\tilde{\psi}(x) &= \langle x h_0' \xi, h_0' \xi \rangle \\ &= \langle x v' t_0 \xi, v' t_0 \xi \rangle = \langle x v'^* v' t_0 \xi, t_0 \xi \rangle \\ &= \langle x v'^* h_0' \xi, t_0 \xi \rangle = \langle x t_0 \xi, t_0 \xi \rangle \\ &= \langle t_0 x t_0 \xi, \xi \rangle \\ &= \tilde{\phi}(t_0 x t_0) \quad \text{for } x \in s(\phi) M s(\phi).\end{aligned}$$

Now we have

$$\begin{aligned}\psi(x) &= \psi(s(\phi) x s(\phi)) = \tilde{\psi}(s(\phi) x s(\phi)) \\ &= \tilde{\phi}(t_0 s(\phi) x s(\phi) t_0) \\ &= \phi(t_0 x t_0) \quad \text{for } x \in M.\end{aligned}$$

This completes the proof.

References

1. J. Dixmier, *Les algèbres d'opérateurs dans l'espace hilbertien*, Gauthier-Villars, Paris, 1957.

2. S. Sakai, *On linear functionals of W*-algebras*, Proc. Japan Acad. **34** (1958), 571–574.

3. ———, *The theory of W*-algebras*, Lecture notes, Yale University, New Haven, Connecticut, 1962.

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