SETS OF UNIQUENESS FOR SOME CLASSES OF TRIGONOMETRICAL SERIES[1]

BY Y. KATZNELSON

Communicated by W. Rudin, May 21, 1964

The group of real numbers modulo 2π is denoted by T. The Fourier coefficients of a distribution μ on T are denoted, as usual, by $\hat{\mu}(n)$. If B is a space of (doubly infinite) sequences we denote by $\mathfrak{F}B$ the space of those distributions μ on T for which $\{\hat{\mu}(n)\}\in B$; and if B is normed we put $\|\mu\|_{\mathfrak{F}B}=\|\{\hat{\mu}(n)\}\|_B$. A closed set E is a set of uniqueness for B if E carries no element of $\mathfrak{F}B$. The purpose of this note is to construct sets of positive measure which are sets of uniqueness for l^p, $1\leqq p<2$.

LEMMA. *Let $\epsilon>0$, $1\leqq p<2$. There exists a closed set $E_{\epsilon,p}\subset T$ having the following properties:*

(1) measure $(E_{\epsilon,p})>2\pi-\epsilon$.

(2) *If μ is carried by $E_{\epsilon,p}$ then* $\|\mu\|_{\mathfrak{F}l^\infty}\leqq\epsilon\|\mu\|_{\mathfrak{F}l^p}$.

PROOF. Let $\gamma>0$. Put

$$f_\gamma(x)=\begin{cases}\dfrac{\gamma-2\pi}{\gamma} & (0<x<\gamma \mod 2\pi),\\[2ex] 1 & (\gamma\leqq x<2\pi \mod 2\pi).\end{cases}$$

Then, by the theorem of Hausdorff-Young (or by direct computation),

$$\|f_\gamma\|_{\mathfrak{F}l^q}\leqq K_p\|f_\gamma\|_{L^p}\sim K_p\gamma^{1/p-1},\quad\text{where}\quad 1/p+1/q=1.$$

Choose an integer N (large) and integers $\lambda_1,\cdots,\lambda_N$ "lacunary" enough so that, taking $\gamma=\epsilon/N$, we have

$$\left\|\frac{1}{N}\sum_1^N f_\gamma(\lambda_j x)\right\|_{\mathfrak{F}l^q}\leqq 2N^{1/q-1}\|f_\gamma\|_{\mathfrak{F}l^q}.$$

Denote

$$F(x)=\frac{1}{N}\sum_1^N f_\gamma(\lambda_j x);$$

then,

[1] This research was partially supported by the United States Air Force Office of Scientific Research under grant No. AFOSR-407-63.

$$\|F\|_{\mathfrak{F}l^q} \leqq 2K_pN^{1/q-1}\gamma^{1/p-1} = 2K_p\epsilon^{1/q-1}\cdot\gamma^{1/p-1/q}$$

so that, if N is large enough $\|F\|_{\mathfrak{F}l^q} < \epsilon$. Put

$$E_{\epsilon,p} = \{x; F(x) = 1\}.$$

Now $F(x)=1 \Leftrightarrow f_\gamma(\lambda_j x)=1$ for all j, so that measure $(E_{\epsilon,p}) \geqq 2\pi - N\gamma = 2\pi - \epsilon$.

Let μ be carried by $E_{\epsilon,p}$, $\mu \in \mathfrak{F}l^p$, then $|\hat{\mu}(n)| = |\int e^{-inx}d\mu| = |\int e^{-inx}F(x)d\mu| = |\sum \hat{F}(m-n)\hat{\mu}(m)| \leqq \|F\|_{\mathfrak{F}l^q}\|\mu\|_{\mathfrak{F}l^p} < \epsilon\|\mu\|_{\mathfrak{F}l^p}$ which proves the lemma.

THEOREM. *There exists a set E of positive measure on T which is a set of uniqueness for $\bigcup_{p<2} l^p$.*

PROOF. Take $\epsilon_n = 10^{-n}$, $p_n = 2 - \epsilon_n$,

$$E = \bigcap_{n=36}^{\infty} E_{\epsilon_n, p_n}.$$

YALE UNIVERSITY

INJECTIVE ENVELOPES OF BANACH SPACES

BY HENRY B. COHEN

Communicated by V. Klee, July 30, 1964

1. **Introduction.** We consider the category whose objects are Banach spaces and whose *maps* are the linear operators of norm not exceeding 1 from one Banach space into another. A Banach space Z is *injective* if it has the same Hahn-Banach extension property that is possessed by the scalars (real or complex); that is, any Z-valued map from a subspace of a Banach space Y extends to a Z-valued map of the same norm on all of Y. An injective envelope of a Banach space B is a pair $(I, \epsilon B)$, ϵB an injective Banach space and $I: B \to \epsilon B$ a linear isometry (our linear isometries need not be onto), such that the only subspace of ϵB that is injective and contains $I[B]$ is ϵB itself. In this note, we demonstrate the existence and uniqueness of the injective envelope of a Banach space and, in the process, we give a short proof of the fact that an injective Banach space is linearly isometric with a function space $C(M)$, M compact Hausdorff and extremally disconnected.