# APPROXIMATION THEOREMS FOR SEMI-GROUP OPERATORS IN INTERMEDIATE SPACES

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Let $X$ be a real or complex Banach-space; if $f\in X$, $\|f\|$ denotes the norm of $f$. If $E(X)$ denotes the Banach-algebra of endomorphisms of $X$, $\{T(t)\}$ is called a one-parameter semi-group in $E(X)$ of class $(C_0)$, if (i) $T(t)\in E(X)$ for $t\in[0, \infty)$, $T(0)=I$ (identity); (ii) $T(t+u) = T(t)T(u)$ for $t, u\in[0, \infty)$; (iii) $\lim_{t\downarrow 0}\|T(t)f-f\|=0$ for all $f\in X$.

Under these hypotheses the infinitesimal operator of $\{T(t)\}$ is a closed linear operator $A$ defined by

$$\lim_{t\downarrow 0}\|t^{-1}[T(t)f-f]-Af\|=0$$

with domain of definition $D(A)$ dense in $X$. $D(A)$ becomes a Banach-space, if the norm is defined by $\|f\|+\|Af\|$ (see E. Hille and R. S. Phillips [3, Chapter X]).

One of the authors [1] has studied the problems of best approximation in semi-group theory. Thus:

*Let $\{T(t)\}$ be a semi-group of class $(C_0)$ defined on $X$.*

(i) *If $\|T(t)f-f\|=o(t)$ $(t\downarrow 0)$, then $Af=\Theta$ and $T(t)f\equiv f$.*

(ii) *For each $f\in D(A)$ we have $\|T(t)f-f\|=O(t)$ $(t\downarrow 0)$.*

(iii) *If $X$ is reflexive and $\|T(t)f-f\|=O(t)$ $(t\downarrow 0)$, then $f\in D(A)$.*

The statements (i) and (ii) go back to E. Hille [3, Chapter X]. For a generalization of this theorem see the cited paper as well as K. de Leeuw [4] and P. L. Butzer and H. G. Tillmann [2].

It is the object of this note to characterize the set of elements $f$, for which the order of approximation of $f$ by $T(t)f$ is not the best possible, i.e., we will not treat saturation problems. In this case, the following general theorem holds.

THEOREM 1. *Let $\{T(t)\}$ be a semi-group of class $(C_0)$, let $T(t)[X]\subset D(A)$ for each $t>0$ and $\|AT(t)\|\leqq M_0t^{-1}$, then*

$$\|T(t)f-f\|=O[\phi(1/t)] \qquad (t\downarrow 0)$$

*implies*

$$\|AT(t)f\|\leqq M_1+M_2t^{-1}\phi(1/t)+M_3\int_1^{1/t}\phi(u)du \qquad (0<t\leqq 1),$$

*where* $\phi(u)$ *is a positive nonincreasing function in* $[1, \infty)$ *and* $M_i$ $(i=0, 1, 2, 3)$ *are constants.*[1]

We may remark that M. Zamansky [7] has established a theorem of this type for trigonometric polynomials.

COROLLARY. *Under the conditions of Theorem* 1 *we have*

(i) $\|T(t)f-f\|=O(t^\alpha)$ $(0<\alpha<1;\ t\downarrow 0)$ *if and only if* $\|AT(t)f\|=O(t^{\alpha-1})$ $(t\downarrow 0)$;

(ii) *if* $\|T(t)f-f\|=O(t)$ $(t\downarrow 0)$, *then* $\|AT(t)f\|=O(\log 1/t)$ $(t\downarrow 0)$.

The corollary is an immediate consequence of the theorem.

**Sketch of proof of Theorem 1.** Setting $t_k=1/2^k$ $(k=0, 1, 2, \cdots)$, we denote by $U_k$ the operator $T(t_k)-T(t_{k-1})$. Then by the semi-group property $AU_kf=AT(t_k)[f-T(t_{k-1})f]-AT(t_{k-1})[f-T(t_k)f]$, and making use of the assumptions one has

$$\begin{aligned}\|AU_kf\| &\leqq \|AT(t_k)\|\,\|f-T(t_{k-1})f\|+\|AT(t_{k-1})\|\,\|f-T(t_k)f\| \\ &\leqq M2^{k-1}\phi(2^{k-1}) \qquad (k=1, 2, \cdots).\end{aligned}$$

Now, let $t$ be given in $(0, 1]$, we choose an integer $n$ such that $t_n<t\leqq t_{n-1}$. Then

$$\|AT(t_n)f-AT(t_{n_0})f\| \leqq \sum_{k=n_0+1}^{n}\|AU_kf\| \leqq 2M\int_{2n_0-1}^{1/t}\phi(u)du.$$

Similarly, we get

$$\|AT(t)f-AT(t_n)f\| \leqq Mt^{-1}\phi(1/t),$$

and, furthermore,

$$\begin{aligned}\|AT(t)f\| \leqq \|AT(t_{n_0})f\| &+ \|AT(t_n)f-AT(t_{n_0})f\| \\ &+ \|AT(t)f-AT(t_n)f\|,\end{aligned}$$

which proves the theorem for $n_0=1$.

As an application we will discuss the singular integral of Abel-Poisson. Let $f$ be a continuous, $2\pi$-periodic function $(f\in C_{2\pi})$, with $\|f\|=\max_x|f(x)|$. Abel's method of summation of the Fourier series of $f$ defines the singular integral

$$\begin{aligned}[V(t)f](x) &= V(f; e^{-t}; x) = a_0/2+\sum_{k=1}^{\infty}(a_k\cos kx+b_k\sin kx)e^{-kt} \\ &= \frac{1}{\pi}\int_{-\pi}^{\pi}f(u)P(e^{-t}; x-u)du \qquad (0<t<\infty),\end{aligned}$$

[1] If $T(t)[X]\subset D(A)$ for each $t>0$, then $AT(t)$ exists as a bounded linear operator on $X$ for $t>0$; $\|AT(t)\|$ denotes the operator norm.

with

$$P(r; u) = \frac{1}{2} \frac{1 - r^2}{1 - 2r \cos u + r^2} \qquad (r = e^{-t}; 0 \leqq r < 1).$$

$V(t)$ is a semi-group of class $(C_0)$ with $\|V(t)\| = 1$. $D(A)$ is the set of functions $f$, for which the derivative of the conjugate function, thus $\tilde{f}' \sim \sum_{k=1}^{\infty} k(a_k \cos kx + b_k \sin kx)$ is an element of $C_{2\pi}$. Furthermore, $V(t)[C_{2\pi}] \subset D(A)$ for all $t > 0$ and

$$\|AV(t)\| = \frac{1}{\pi} \int_{-\pi}^{\pi} | Q'(e^{-t}; u) | \, du \leqq 4t^{-1} \qquad (0 < t < \infty),$$

whereby

$$Q(r; u) = \frac{r \sin u}{1 - 2r \cos u + r^2}.$$

Now, with the aid of the corollary we have

THEOREM 2. *Let $f \in C_{2\pi}$, and let $V(f; r; x)$ be the Abel-Poisson integral. The following statements are equivalent if* $0 < \alpha < 1$:
(i) $\|f(x+h) - f(x)\| = O(|h|^{\alpha})$ $(h \to 0)$;
(ii) $\|f(x+h) - 2f(x) + f(x-h)\| = O(|h|^{\alpha})$ $(h \to 0)$;
(iii) $\|\tilde{V}'(f; r; x)\| = O(1-r)^{\alpha-1}$ $(r \uparrow 1)$;
(iv) $\|V''(f; r; x)\| = O(1-r)^{\alpha-2}$ $(r \uparrow 1)$;
(v) $\|V(f; r; x) - f(x)\| = O(1-r)^{\alpha}$ $(r \uparrow 1)$.

The equivalence of the statements (i)-(iv) above is known, the results being mainly due to G. H. Hardy and J. E. Littlewood (see A. Zygmund [8, Chapter VII]). These proofs, in contrast to ours, used complex methods. The fact that (v) is equivalent to (i) is a new contribution.

In some of his papers, J. L. Lions [5] has studied trace theorems and theorems of interpolation in semi-group theory. He introduced the so-called intermediate spaces $X[p, \alpha, A]$: Let $\{T(t)\}$ be a semi-group of class $(C_0)$ with $\|T(t)\| \leqq M_0$ for all $t \geqq 0$. We denote by $X[p, \alpha, A]$ the set of elements $f \in X$ for which the integral

$$\int_0^{\infty} t^{(\alpha-1)p} \|T(t)f - f\|^p dt$$

exists, where $-1/p < \alpha < 1 - 1/p$, $1 \leqq p \leqq \infty$. $X[p, \alpha, A]$ becomes a Banach-space under the norm

$$\|f\| + \left\{ \int_0^\infty t^{(\alpha-1)p} \|T(t)f - f\|^p dt \right\}^{1/p}.$$

(If $p = \infty$, the modification is evident.) It is easy to see that

$$D(A) \subset X[p, \alpha, A] \subset X.$$

With methods stated above we can prove the following theorem concerning $X[p, \alpha, A]$.

THEOREM 3. *Let* $\{T(t)\}$ *be a semi-group of class* $(C_0)$, *let* $\|T(t)\| \leqq M_0$, $T(t)[X] \subset D(A)$ *for each* $t > 0$ *and* $\|AT(t)\| \leqq M_1 t^{-1}$, *then* $f \in X[p, \alpha, A]$ *if and only if the integral*

$$\left\{ \int_0^\infty t^{\alpha p} \|AT(t)f\|^p dt \right\}^{1/p}$$

*is finite.*

By use of this theorem one may infer some of the results due to M. H. Taibleson [6] for the singular integral of Poisson-Cauchy in $n$-dimensional Euclidean space, since this integral is a semi-group operator satisfying the conditions of Theorem 3.

The proofs of these and further results will appear elsewhere.

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