A TWO-DIMENSIONAL SINGULAR INTEGRAL EQUATION OF DIFFRACTION THEORY[1]

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The formulation of a problem in diffraction theory has led us to consider the two-dimensional singular integral equation

$$\iint_{Q_{13}} f(t_1, t_2)k(|x_1 - t_1|, |x_2 - t_2|)dt_1dt_2 = 0 \tag{1}$$

where: Q_{13} denotes the union of quadrants I, III; f is unknown, but must vanish on quadrants II, IV; the equation is valid only for $x = (x_1, x_2)$ in Q_{13}; and k is the diffraction-theoretic kernel

$$k(x) = -(4\pi r)^{-1}\exp(-i\beta r) \tag{2}$$

with $r = +(x_1^2 + x_2^2)^{1/2}$ and β complex [$\mathrm{Im}(\beta) < 0$].

In earlier physical investigations, we had encountered variants of (1) in which the domains of integration and validity were (a) two contiguous quadrants (see [4]) or (b) one quadrant (see [5], [7]); and it is clear that the equation over three quadrants may be treated by methods applicable to the complementary case (b). Thus, the present study of (1) completes a theory of two-dimensional convolution-type equations with the diffraction-theoretic kernel k over quadrants of the x_1x_2-plane. Since these equations generalize the one-dimensional convolution-type on the half-line (i.e., the classical equation of Wiener and Hopf [9]), the theory is a partial extension of Wiener and Hopf's ideas from one to two dimensions.

Our analysis may be divided into three parts:

I. **Preparatory.** The integral equation (1) is extended to X, the whole x_1x_2-plane, whereupon the left side becomes a convolution (Wiener's "Faltung") while the right side $h(x)$ is defined (but not known) on $X - Q_{13}$, and $h = 0$ on Q_{13}:

$$f * k = h. \tag{3}$$

The two-dimensional Laplace transformation ([1], Chapter VI of [2], or [3]) then maps (3) into the transform equation (capital letters denote transforms; $w = (w_1, w_2)$ denotes a point in a product-space

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of two complex variables, with $w_j = u_j + iv_j$ and $j = 1, 2$ here and in what follows):

$$F(w)K(w) = H(w) \tag{4}$$

which is to be solved for the two unknown functions F, H.

It is known that $K(w) = (i/2)(w_1^2 + w_2^2 + \beta^2)^{-1/2}$; thus, K is analytic for $u = (u_1, u_2)$ in a product domain B: $W_1 \times W_2$, where the W_j are vertical strips interior to $|u_j| \leqq |\mathrm{Im}(\beta)|$. Assume next that F is a distribution (cf. [3, Proposition 4.2, p. 14]) representable as

$$F(w) = P(w)G(w), \tag{5}$$

where P is a polynomial, and

$$G(w) = (\mathfrak{L}_1 + \mathfrak{L}_3)g(x), \tag{6}$$

the restricted Laplace transform $\mathfrak{L}_n$ being defined by an integral over the closed nth quadrant:

$$\mathfrak{L}_n g = \iint_{Q_n} g(x) \exp(-w \cdot x) dx_1 dx_2 \tag{7}$$

($w \cdot x = w_1x_1 + w_2x_2$). Finally, let $g(x) \exp(-w \cdot x)$ be of bounded L_2 norm over Q_1, Q_3 for u in the respective domains

$$C_1\colon u_j > b_j > 0, \tag{8.1}$$

$$C_3\colon u_j < -b_j < 0 \tag{8.2}$$

with $C_1 \cap C_3$ empty, as indicated in (8.1), (8.2), while $C_1 \cap B$ and $C_3 \cap B$ are nonempty.

[REMARK. If $C_1 \cap C_3$ is nonempty, $G \equiv 0$. The same situation is noted in [8], whose subject is the presently relevant one of characterizing two-variable Laplace transforms of functions with support in Q_{13}. Some of the considerations which arise are exemplified in the proof of Lemma II.2 below.] It may then be shown that:

STATEMENT I.1. $F(w)$ and $H(w)$ as well as $K(w)$ are analytic for u in B.

STATEMENT I.2. $F(w) = F_1(w) + F_3(w)$, $H(w) = H_2(w) + H_4(w)$, where: subscripts n ($n = 1, 2, 3, 4$) denote functions analytic for u in (B, n), while (B, n) signifies the convex closure of B and the nth quadrant of the u-plane.

II. Factorization is the key step as in [9], but the single factorization lemma of Wiener and Hopf's one-dimensional theory must now be replaced by two lemmas:

LEMMA II.1. *$K(w)$ may be uniquely expressed as the product of four* *functions* $M_n(w)$, *analytic and nonzero for* u *in* (B, n). [*This is shown, and the* $M_n(w)$ *are explicitly calculated, in* [5, §5].]

LEMMA II.2. $K_{13}(w) = M_1(w)M_3(w)$ *and* $K_{24}(w) = M_2(w)M_4(w)$ *are analytic and nonzero in the respective pairs of disjoint u-domains* ($-\infty < v_j < +\infty$ *throughout*) q_1, q_3 *and* q_2, q_4, *where we have* (*for any* $\delta > 0$)

$$q_1: (u_1 + u_2) \geqq (|\operatorname{Im}(\beta)| + \delta) \qquad (u_j > 0) \tag{9}$$

and q_2, q_3, q_4 *are successive reflections of* q_1 *in* $u_2 > 0$, $u_1 < 0$, $u_2 < 0$.

PROOF. Introduce the function

$$\begin{aligned} \phi(x) &= N_0(\beta r), \qquad && x \in Q_{13} \\ &= 0, && x \in Q_{24} \end{aligned} \tag{10}$$

where $N_0(\beta r)$ is the Bessel function of the second kind (Neumann's function). The image of ϕ under two-dimensional Laplace transformation is, as shown in [6],

$$\Phi(w) = 2i(w_1^2 + w_2^2 + \beta^2)^{-1}[-i + w_1 s_2 S_2 + w_2 s_1 S_1], \tag{11}$$

with

$$s_j = (w_j^2 + \beta^2)^{-1/2} \qquad (s_j = \beta^{-1} \text{ at } w_j = 0) \tag{11.1}$$

$$S_j = (2i\pi^{-1}) \log [\beta^{-1}(w_j + s_j^{-1})] \tag{11.2}$$

and, significantly,

$$\mathcal{L}\phi(x) = 2\mathcal{L}_1\phi(x) = 2\mathcal{L}_3\phi(x). \tag{12}$$

As appears from (12), $\Phi(w) \equiv \mathcal{L}\phi(x)$ is analytic for u in q_1 *and* for u in q_3. The same is true of $K_{13}(w)$, since it may be shown (by the reasoning of [5, §5]) that

$$K_{13}(w) = \exp\left[-\int^{\beta} \Phi(w; \beta)d\beta\right] \tag{13}$$

where Φ is written $\Phi(w; \beta)$ to emphasize the dependence on β, and it is understood that β is allowed to vary in a small neighborhood of its fixed value for purposes of the integration. The assertion for K_{13} is therefore proved, and the proof for K_{24} follows by symmetry.

III. **Solutions of the transform equation and the integral equation.** Two theorems may now be proved without difficulty (the first requires only verification):

THEOREM III.1. *The transform equation* (4) *has the solutions*

$$F(w) = c_0[K_{13}(w)]^{-1}, \tag{14.1}$$

$$H(w) = c_0 K_{24}(w) \tag{14.2}$$

where c_0 *is an arbitrary constant* [*the same arbitrary constant in* (14.1), (14.2)].

THEOREM III.2. *The functions* F, H *of* (14.1), (14.2) *possess two-variable Laplace inverses* $f(x)$, $h(x)$, *and the latter pair are literal solutions of* (3). [*The function* $f(x)$ *is of course a literal solution of* (1), *as well as of the extended equation* (3).]

REFERENCES

1. S. Bochner, *Bounded analytic functions in several variables and multiple Laplace integrals*, Amer. J. Math. **59** (1937), 732–738.
2. S. Bochner and W. T. Martin, *Several complex variables*, Princeton Univ. Press, Princeton, N. J., 1948.
3. J. Leray, *Hyperbolic differential equations*, lecture notes, Inst. for Advanced Study, Princeton, N. J., 1954.
4. J. Radlow, *Diffraction of a dipole field by a unidirectionally conducting semi-infinite screen*, Quart. Appl. Math. **17** (1959), 113–127.
5. ———, *Diffraction by a quarter-plane*, Arch. Rational Mech. Anal. **8** (1961), 139–158.
6. ———, *On the double Laplace transforms of some Green's functions*, J. Math. and Phys. **38** (1959), 203–205.
7. ———, *Diffraction by a right-angled dielectric wedge*, Internat. J. Eng. Sci. (to appear).
8. ———, *A two-dimensional version of Laplace's integral equation*, Abstract 603-141, Notices Amer. Math. Soc. **10** (1963), 474.
9. N. Wiener and E. Hopf, *Über eine Klasse Singulärer Integralgleichungen*, S.-B. Berlin Akad. Wiss. (1931), 696–706.

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