ON BOUNDING HARMONIC FUNCTIONS BY LINEAR INTERPOLATION

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It is well known [1], [4] that Poisson's formula for the value at the origin O of a function which is harmonic inside a circle $(x-x_0)^2+(y-y_0)^2=A^2$ can be written in the form

$$u(O)=\frac{1}{2\pi}\int_0^{2\pi}\frac{R(\theta+\pi)u(R(\theta),\theta)+R(\theta+\pi)u(R(\theta+\pi),\theta+\pi)}{R(\theta)+R(\theta+\pi)}\,d\theta,$$

where $r=R(\theta)$ is the polar equation of the boundary. Thus the value of a harmonic function at any point in a circle is an average of the values obtained by linear interpolation of the boundary values at the ends of each chord through the point.

In particular, it follows that

$$u(O)\leqq\max\frac{R(\theta+\pi)u(R(\theta),\theta)+R(\theta)u(R(\theta+\pi),\theta+\pi)}{R(\theta)+R(\theta+\pi)}\cdot$$

It is tempting to conjecture that a similar inequality holds for harmonic functions in any convex or even star-shaped domain. Recently J. Barta [2], [3] has given two incomplete proofs of this conjecture.

We shall show that in general no inequality of the form

$$u(O)\leqq M\max\frac{R(\theta+\pi)u(R(\theta),\theta)+R(\theta)u(R(\theta+\pi),\theta+\pi)}{R(\theta)+R(\theta+\pi)}\tag{1}$$

can hold for all harmonic functions in a star-shaped domain $r<R(\theta)$. In fact, an inequality of the form (1) holds for each point O of a convex domain D only if D is the interior of a circle.

We first prove:

LEMMA. *Let G be the Green's function with singularity at O for the two-dimensional domain D: $r<R(\theta)$. An inequality of the form* (1) *holds for all harmonic functions u if and only if the identity*

$$\begin{aligned}&R(\theta)(R(\theta)^2+R'(\theta)^2)^{1/2}\frac{\partial G}{\partial n}(R(\theta),\theta)\\&\quad=R(\theta+\pi)(R(\theta+\pi)^2+R'(\theta+\pi)^2)^{1/2}\frac{\partial G}{\partial n}(R(\theta+\pi),\theta+\pi)\end{aligned}\tag{2}$$

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holds for all θ. *If* (2) *is satisfied*, (1) *holds with* $M=1$.

PROOF. Let a be a constant such that

$$\frac{R(\theta+\pi)u(R(\theta),\theta)+R(\theta)u(R(\theta+\pi),\theta+\pi)}{R(\theta)+R(\theta+\pi)} \leqq a \tag{3}$$

for all θ.

We write the representation

$$u(O) = -\oint_{r=R(\theta)} u \frac{\partial G}{\partial n} ds$$

in the form

$$u(O) = a - \int_0^{2\pi} [u(R(\theta),\theta) - a] \frac{\partial G}{\partial n}(R(\theta),\theta)\frac{ds}{d\theta} d\theta. \tag{4}$$

In the identity

$$\int_0^{2\pi} f(\theta)g(\theta)d\theta = \frac{1}{2}\int_0^{\pi} \{[f(\theta)+f(\theta+\pi)][g(\theta)+g(\theta+\pi)] + [f(\theta)-f(\theta+\pi)][g(\theta)-g(\theta+\pi)]\} d\theta$$

we let

$$f(\theta) = \frac{u(R(\theta),\theta) - a}{R(\theta)}, \tag{5}$$

$$\begin{aligned} g(\theta) &= -R(\theta)\frac{\partial G}{\partial n}(R(\theta),\theta)\frac{ds}{d\theta} \\ &= -R(\theta)(R(\theta)^2 + R'(\theta)^2)^{1/2}\frac{\partial G}{\partial n}(R(\theta),\theta). \end{aligned} \tag{6}$$

Then $g(\theta) \geqq 0$. By (3),

$$f(\theta) + f(\theta+\pi) \leqq 0. \tag{7}$$

Thus,

$$u(O) \leqq a + \frac{1}{2}\int_0^{\pi} [f(\theta) - f(\theta+\pi)][g(\theta) - g(\theta+\pi)]d\theta. \tag{8}$$

Equality holds if $f(\theta)+f(\theta+\pi)\equiv 0$; that is, for those boundary values $u(R(\theta),\theta)$ satisfying

$$R(\theta+\pi)u(R(\theta),\theta) + R(\theta)u(R(\theta+\pi),\theta+\pi) = a[R(\theta)+R(\theta+\pi)].$$

If $f(\theta)$ is made to satisfy only this condition, the function $f(\theta)-f(\theta+\pi)$ is completely arbitrary for $0\leqq\theta<\pi$. The right-hand side of (8) and therefore also $u(O)$ can be made arbitrarily large unless $g(\theta)-g(\theta+\pi)=0$. This is the condition (2).

If (2) is satisfied, (8) becomes $u(O)\leqq a$, which is (1) with $M=1$.

We remark that (2) is certainly satisfied if the symmetry condition

$$R(\theta+\pi)=R(\theta)$$

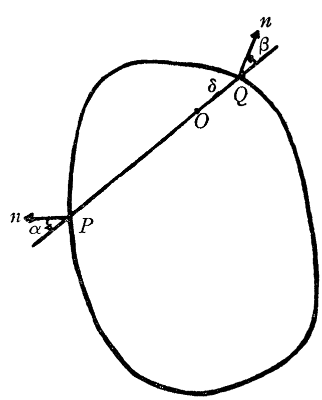
holds. This means that the point O bisects each chord through it. This is true at the center of an ellipse, or of a parallelogram. In such a case we find that

$$u(O)\leqq \max \tfrac{1}{2}[u(R(\theta),\theta)+u(R(\theta+\pi),\theta+\pi)].$$

We can now prove:

THEOREM. *If a bound of the form* (1) *for harmonic functions u holds at each point O of a convex domain D with smooth boundary C, then C is a circle.*

PROOF. We consider the chord PQ connecting any two boundary points P and Q. Let its length be d, and let O be the point on this chord at distance δ from Q.



Let the chord make angles α and β, respectively, with the normals at P and Q.

By hypothesis, (1) holds at O. Hence by the lemma we have

$$\frac{(d-\delta)^2}{\cos\alpha}\frac{\partial G}{\partial n}(O,P)=\frac{\delta^2}{\cos\beta}\frac{\partial G}{\partial n}(O,Q). \tag{9}$$

We let O approach Q by making $\delta\to 0$. It is easily seen that

$$\frac{\partial G}{\partial n}(O, Q) = \frac{-1}{\pi\delta}\cos\beta + O(1).$$

(The leading term comes from Green's function for the half-plane.) On the other hand, since $(\partial G/\partial n)(O, P) = 0$ for O on C,

$$\frac{\partial G}{\partial n}(O, P) = -\cos\beta \frac{\partial^2 G}{\partial n_P \partial n_Q}(P, Q) + O(\delta^2).$$

Dividing (9) by δ and letting $\delta \to 0$, we find

$$d^2 \frac{\partial^2 G}{\partial n_P \partial n_Q}(P, Q) \frac{\cos\beta}{\cos\alpha} = \frac{1}{\pi}\cdot$$

The function $\partial^2 G/\partial n_P \partial n_Q$ is symmetric in P and Q. Letting $\delta \to d$, we find the same equation with α and β interchanged. Hence $\cos\alpha = \cos\beta$. This is true for all P and Q on C. Letting $Q \to P$ on C and using the fact that β is a continuous function of Q, we find that $\alpha = \beta$.

An elementary exercise in differential geometry shows that $\alpha = \beta$ for all P and Q on C implies that C is a circle. This proves the theorem.

REMARK. If we restrict our attention to non-negative u:

$$u(R(\theta), \theta) \geqq 0,$$

the inequality (8) does lead to a bound of the form (1) with the best possible constant

$$M = 1 + \int_0^\pi \max\left\{\frac{1}{R(\theta+\pi)}[g(\theta) - g(\theta+\pi)], \frac{1}{R(\theta)}[g(\theta+\pi) - g(\theta)]\right\} d\theta. \tag{10}$$

However, the evaluation of this constant requires rather detailed information about the kernel $\partial G/\partial n$, which is difficult to come by.

In this case the maximum principle gives (1) with

$$M = 1 + \max_{0 \leqq \theta \leqq 2\pi} \left\{\frac{R(\theta+\pi)}{R(\theta)}\right\},$$

which is just what one obtains by means of crude estimates for the Green's function in (10).

The analogous results in n dimensions can be proved in the same manner.

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A NOTE ON THE FUNDAMENTAL THEORY OF ORDINARY DIFFERENTIAL EQUATIONS

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In this note we present some results on various problems connected with ordinary differential equations which do not necessarily satisfy a uniqueness condition. Using the concept of an integral funnel we are able to generalize the classical theorem on continuity with respect to initial conditions. This then leads to a reformulation of the problem of classifying the solutions of a given differential equation. That is, it is shown that every continuous vector field $f(x)$ on W gives rise to a bicontinuous injection of W into a space of functions $\boldsymbol{H}$, and consequently the problem of classifying solutions is equivalent to the problem of characterizing this family of bicontinuous injections. A detailed discussion, with proofs, will appear later.

1. Introduction. Let us consider the differential equation

$$x' = f(x) \tag{1}$$

where f is defined and continuous on some open, connected set W in R^n, real n-space. We shall let $W^* = W \cup \{\omega\}$ denote the one-point compactification of W. There is then at least one solution $\phi(p, t)$ of (1) through every point $p \in W$ with $\phi(p, 0) = p$. Moreover, every solution is defined on some maximal interval J_p where either $J_p = R^1$ or $\phi(p, t) \to \{\omega\}$ as $t \to bdy\ J_p$. It should be noted that since the solutions of (1) may not be unique, the interval J_p depends not only on p

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