A COHOMOLOGY THEORY BASED UPON SELF-CONJUGACIES OF COMPLEX VECTOR BUNDLES[1]

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Let $\xi: E \rightarrow X$ be a complex n-plane bundle. A self-conjugacy of ξ is a map $\chi: E \rightarrow E$ which is a norm-preserving bundle equivalence on the underlying $2n$-dimensional real bundle and has the property that $\chi(\lambda e) = \bar{\lambda}\chi(e)$ for $e \in E$ and λ complex.

If χ_i is a self-conjugacy of ξ_i, define (ξ_1, χ_1) to be equivalent to (ξ_2, χ_2) if there is a bundle equivalence $\mu: E_1 \rightarrow E_2$ with $\mu^{-1}\chi_2\mu$ homotopic to χ_1; i.e. there is a continuous map $H: E_1 \times I \rightarrow E_1$ with $H|E_1 \times 0 = \mu^{-1}\chi_2\mu$, $H|E_1 \times 1 = \chi_1$, and $H|E_1 \times t$ is a self-conjugacy of ξ_1 for each $t \in I$.

If χ is a self-conjugacy of ξ, χ may be interpreted as a cross-section of the bundle ξ' associated to ξ, whose fibre is the space of 1-1 norm-preserving self-conjugacies of the fibre of ξ. The fibre of ξ', denoted $V(n)$, is homeomorphic to $U(n)$, the unitary group, but there is no canonical homeomorphism. A noncanonical homeomorphism is given by left or right composition with any element of $V(n)$. Let $W(n)$ be the total space of the $V(n)$ bundle associated to the universal bundle over $BU(n)$. Then if $f: X \rightarrow BU(n)$ is a classifying map for ξ and χ is a self-conjugacy of ξ, χ interpreted as a cross-section of ξ' gives a lifting of f to a map $g: X \rightarrow W(n)$. Conversely any such lifting of f gives a self-conjugacy of ξ.

THEOREM I. *This construction defines a* 1-1 *correspondence between homotopy classes of maps* $X \rightarrow W(n)$ *and equivalence classes of bundles with self-conjugacy with* X *as base space.*

There is no difficulty about extending Whitney sums to bundles with self-conjugacy. One writes:

$$(\chi_1 \oplus \chi_2)(e_1, e_2) = (\chi_1 e_1, \chi_2 e_2)$$

verifying easily that if χ_i is a self-conjugacy of ξ_i, then $\chi_1 \oplus \chi_2$ is a self-conjugacy of $\xi_1 \oplus \xi_2$.

The definition of Whitney sum gives rise to a definition of stable equivalence: two bundles with self-conjugacy are stably equivalent if their sums with some third bundle with self-conjugacy are equivalent. A classifying space for stable equivalence classes is $Z \times W$, where Z

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denotes the integers and W may be described as follows:

Let $C^{(\omega)}$ be an infinite-dimensional complex vector space with the orthonormal basis $\{\alpha_i\}$, $-\infty < i < +\infty$. Let ϕ be the self-conjugacy of $C^{(\omega)}$ given by $\phi(\sum \lambda_i \alpha_i) = \sum \bar{\lambda}_i \alpha_i$. For each n let $L_{(n)}$ be the subspace spanned by $\{\alpha_i\}$, $i \leqq n$, and $L^{(n)}$ the subspace spanned by $\{\alpha_i\}$, $i > n$. A point of BU is the span of $L_{(n)}$ and an n-dimensional subspace of $L^{(n)}$ for any $n \leqq 0$. A point of W is a self-conjugacy of a point of BU whose restriction to $L_{(k)}$ coincides with $\phi | L_{(k)}$ for some k. Choose $L_{(0)}$ as a base point for BU and $\phi | L_{(0)}$ as a base point for W. W is a fibre bundle over BU whose fibre V is the space of self-conjugacies of $L_{(0)}$ which agree with ϕ on some $L_{(k)}$. The universal bundle over BU with total space EU and fibre U may also be described in this context. A point of EU is a unitary map of $L_{(0)}$ onto some other point of BU, coinciding with the identity on some $L_{(k)}$ common to both. U is the space of unitary automorphisms of $L_{(0)}$ restricting to the identity on some $L_{(k)}$.

Let $\partial: \pi_n(BU) \to \pi_{n-1}(V)$ be the boundary operator of the homotopy sequence of the fibering, $V \to W \to BU$. Let $\Delta: \pi_n(BU) \to \pi_{n-1}(U)$ be the boundary isomorphism of the universal bundle. Let $\phi': BU \to BU$ be the restriction to BU of the involution which takes each subspace of $C^{(\omega)}$ to its image under ϕ. Let $\phi'': V \to U$ be the homeomorphism induced by right composition with ϕ.

LEMMA I.

$$\phi''_* \partial = \Delta(\phi'^{-1}_*): \pi_n(BU) \to \pi_{n-1}(U).$$

LEMMA II.

$$\phi'_*: \pi_n(BU) \to \pi_n(BU) = \begin{cases} -1 & \text{when } n \equiv 2 \ (4) \\ +1 & \text{when } n \equiv 0 \ (4). \end{cases}$$

THEOREM II.

$$\pi_n(W \times Z) = \begin{cases} Z_2 & \text{when } n \equiv 1 \ (4) \\ 0 & \text{when } n \equiv 2 \ (4) \\ Z & \text{when } n \equiv 3, 4 \ (4). \end{cases}$$

An alternative description of W (up to weak homotopy type) is the following: A point of W is a path $w: I \to BU$ such that $w(1) = \phi'(w(0))$. Let $\tau: BU \wedge BU \to BU$ be the map corresponding to the reduced tensor product [**3**, p. 297].

LEMMA III. *There exists a map* $\theta: (BU \wedge BU) \times I \to BU$ *such that* $\theta(x, y, 0) = \tau(\phi'(x), \phi'(y))$ *and* $\theta(x, y, 1) = \phi'(\tau(x, y))$.

Define τ': $W \wedge W \to W$ by

$$\tau'(w_1, w_2)(s) = \tau(w_1(2s), w_2(2s)), \qquad 0 \leqq s \leqq 1/2,$$
$$\tau'(w_1, w_2)(s) = \theta(w_1(0), w_2(0), 2s - 1), \qquad 1/2 \leqq s \leqq 1.$$

From these definitions and from the Bott periodicity theorem for the unitary group as stated in [1] and [2], follows

THEOREM III. *Let the homotopy class of* σ: $S^4 \to W$ *generate* $\pi_4(W)$. *Let* σ': $\Sigma^4 W \to W$ *be defined by* $\sigma'(s, w) = \tau'(\sigma(s), w)$. *Let* σ'': $W \to \Omega^4 W$ *be adjoint to* σ'. *Then* σ'' *is a weak homotopy equivalence between* W *and the connected component of the base point in* $\Omega^4 W$.

Hence W defines an Ω-spectrum and the construction of [4] gives a periodic cohomology theory, KSC^*. $KSC^0(X)$ is the stable equivalence classes of bundles over X with self-conjugacy.

A more detailed treatment of this material is contained in the author's thesis. The proof of Theorem I is a routine exercise in writing down homotopies. Theorem III depends on the fact that $\Omega^4 W$ is fibered by $\Omega^5 BU$ over $\Omega^4 BU$. σ'' is a fibre preserving map which induces homotopy isomorphisms on the fibre and base space.

This cohomology theory has been developed independently by Donald Anderson of the University of California at Berkeley, who has used it to obtain information about $KO^*(B_G)$, where G is a compact Lie group.

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