HARMONIC ANALYSIS AND THE THEORY OF COCHAINS

VICTOR L. SHAPIRO[1]

1. Let E^2 represent the plane endowed with the usual Cartesian coordinate system, and let R be an open set contained in E^2. We say that X is a 1-cochain defined in R (see [7, p. 5]) if (a) $X(\sigma)$ is a real number for every 1-simplex σ (i.e., oriented line segment) contained in R, (b) $X(-\sigma)=-X(\sigma)$ for every 1-simplex σ contained in R, (c) $X(\sigma)=X(\sigma_1)+\cdots+X(\sigma_n)$ for $\sigma=\sigma_1+\cdots+\sigma_n$ with $\sigma, \sigma_1, \cdots, \sigma_n$ collinear, similarly oriented, and contained in R. X is then extended by linearity to all chains in R; so in particular if τ is a 2-simplex (i.e., oriented triangle), $X(\partial\tau)$ is defined.

We shall call the 1-cochain X a local L^1 1-cochain in R if the following two conditions are met:

(1) there exist two non-negative functions $g_1(x)$ and $g_2(y)$, each locally in L^1 on R such that

(α) if σ is a 1-simplex in R parallel to and oriented like the x-axis, $|X(\sigma)| \leqq \int_\sigma g_1(x)dx$,

(β) if σ is a 1-simplex in R parallel to and oriented like the y-axis, $|X(\sigma)| \leqq \int_\sigma g_2(y)dy$;

(2) there exists a non-negative function $H(x, y)$ locally in L^1 on R such that if τ is a 2-simplex oriented like E^2 with two edges parallel to the x and y-axes and τ is in R, then

$$|X(\partial\tau)| \leqq \int_\tau H(x, y)dxdy.$$

Let Q be a measurable set contained in R with the property that $|R-Q|_2=0$ (where $|\ \ |_j$ represents j-dimensional Lebesgue measure). Using the notation of [7, p. 262], we say that the 1-simplex σ in R is Q-good if $|\sigma-(\sigma\cap Q)|_1=0$. We say that a 2-simplex τ contained in R is Q-excellent if each of the 1-simplices in $\partial\tau$ are Q-good.

We shall call the differential form $\omega(x, y)=a(x, y)dx+b(x, y)dy$ a local L^1 differential 1-form in R if the following three conditions are met:

(3) $a(x, y)$ and $b(x, y)$ are measurable functions in R;

(4) there exists a measurable set $Q\subset R$ with $|R-Q|_2=0$ and two

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non-negative functions $g_1(x)$ and $g_2(y)$ each locally in L^1 on R such that

(α) for (x, y) in Q, $|a(x, y)| \leqq g_1(x)$ and $|b(x, y)| \leqq g_2(y)$,

(β) for every Q-good 1-simplex σ in R, $a(x, y)$ and $b(x, y)$ are measurable functions on σ;

(5) with Q as in (4), there exists a non-negative function $H(x, y)$ which is locally in L^1 on R such that if τ is a Q-excellent 2-simplex in R oriented like E^2 with two edges parallel to the x and y-axes, then

$$\left| \int_{\partial\tau} \omega \right| \leqq \int_{\tau} H(x, y)dxdy.$$

We shall say two differential forms $\omega(x, y) = a(x, y)dx + b(x, y)dy$ and $\omega_1(x, y) = a_1(x, y)dx + b_1(x, y)dy$ are equivalent in R if $a(x, y) = a_1(x, y)$ and $b(x, y) = b_1(x, y)$ almost everywhere in R.

Using harmonic analysis as we have before in similar situations (see [**4**], [**5**], and [**6**]), we shall establish the following theorem relating local L^1 1-cochains and local L^1 differential 1-forms:

THEOREM. *There is a* 1-1 *correspondence between local L^1* 1*-cochains in R and equivalence classes of local L^1 differential* 1*-forms in R in the following sense*:

(a) *Given X a local L^1* 1*-cochain in R, there exists a local L^1 differential* 1*-form in R, ω, and a set Q satisfying* (4) *and* (5) *with respect to ω such that $X(\sigma) = \int_\sigma \omega$ for every Q-good* 1*-simplex σ. Furthermore, ω is unique up to equivalence.*

(b) *Given ω a local L^1 differential* 1*-form in R and a set Q satisfying* (4) *and* (5) *with respect to ω, there exists a local L^1* 1*-cochain in R, X, such that $X(\sigma) = \int_\sigma \omega$ for every Q-good* 1*-simplex σ. Furthermore, if ω_1 is equivalent to ω, it gives rise to the same local L^1* 1*-cochain X.*

Since every flat 1-cochain in R (see [**7**, p. 6]) is a local L^1 1-cochain in R and since every flat differential 1-form in R (see [**7**, p. 263]) is a local L^1 differential 1-form in R, the theorem given above is an extension of Wolfe's theorem [**7**, p. 253 and p. 265] for the special case of 1-cochains defined in open sets of the plane.

2. To establish part (a) of the theorem, we shall suppose from the start that if g_1 and g_2 are the given functions satisfying (1) with respect to X and if g_1 is defined in a neighborhood of x and g_2 is defined in a neighborhood of y, then

$$(6) \quad g_1(x) = \limsup_{|h| \to 0} h^{-1} \int_0^h g_1(x+t)dt, \quad g_2(y) = \limsup_{|h| \to 0} h^{-1} \int_0^h g_2(y+t)dt.$$

This would only alter g_1 and g_2 on a set of 1-dimensional measure zero but would not affect the relationship (1).

Letting $\sigma^1(x, y; h)$ and $\sigma^2(x, y; h)$ be respectively the 1-simplex whose ordered end points are (x, y), $(x+h, y)$ and (x, y), $(x, y+h)$, we define for (x, y) in R and j integer-valued,

$$\begin{aligned} a(x, y) &= \limsup_{|j|\to\infty} jX[\sigma^1(x, y; j^{-1})] \text{ and} \\ b(x, y) &= \limsup_{|j|\to\infty} jX[\sigma^2(x, y; j^{-1})]. \end{aligned} \tag{7}$$

It follows immediately from (1), (6), and (7) that

$$\text{for } (x, y) \text{ in } R,\ |a(x, y)| \leqq g_1(x) \text{ and } |b(x, y)| \leqq g_2(y). \tag{8}$$

We next prove that $a(x, y)$ and $b(x, y)$ are measurable functions in R. In order to establish this fact, we need only show that if S is a closed square with sides parallel to the x and y-axes which is contained in R, then both functions are measurable in S. Consequently, with S given, choose S' a closed square such that $S \subset S'^0 \subset S' \subset R$ where S'^0 designates the interior of S'. Next choose a positive integer j_0 such that for $|j| \geqq j_0$ and for (x, y) in S, $\sigma^1(x, y; j^{-1})$ and $\sigma^2(x, y; j^{-1})$ are both in S'. To establish the measurability of $a(x, y)$ and $b(x, y)$, it is sufficient to show that for a fixed j with $|j| \geqq j_0$, $X[\sigma^1(x, y; j^{-1})]$ and $X[\sigma^2(x, y; j^{-1})]$ are continuous functions when restricted to S.

We now show that $X[\sigma^1(x, y; j^{-1})]$ is a continuous function when restricted to S with a similar proof prevailing to show that $X[\sigma^2(x, y; j^{-1})]$ is a continuous function when restricted to S.

Let (x_0, y_0) be a fixed point in S and let $\epsilon > 0$ be given. Choose δ_1 with $0 < \delta_1 < 1$ such that if σ is a 1-simplex in S' parallel to the x-axis and $|\sigma|_1 < \delta_1$, then $|\int_\sigma g_1(x)dx| < \epsilon$. Similarly, choose δ_2 with $0 < \delta_2 < 1$ such that if σ is a 1-simplex in S' parallel to the y-axis and $|\sigma|_1 < \delta_2$, then $|\int_\sigma g_2(y)dy| < \epsilon$. Next, with $H(x, y)$ given by (2), choose δ_3 with $0 < \delta_3 < 1$ such that if τ is a 2-simplex as described in (2) which is contained in S' (i.e., a right triangle with legs parallel to the x and y-axes and oriented like E^2) and $|\tau|_2 < \delta_3$, then $\int_\tau H(x, y)dxdy < \epsilon$. Set $\delta_4 = \min(\delta_1, \delta_2, \delta_3)$. It then follows immediately from the fact that X is a 1-cochain which meets conditions (1) and (2) that if σ is an arbitrary 1-simplex in S' and $|\sigma|_1 < \delta_4$, then $|X(\sigma)| \leqq 3\epsilon$.

Let N be the first integer greater than the length of the side of S'. Set $\delta = \delta_4/N$, and suppose that (x, y) is in S and that the distance between (x_0, y_0) and (x, y) is less than δ. Then on considering the parallelogram determined by $\sigma^1(x_0, y_0; j^{-1})$ and $\sigma^1(x, y; j^{-1})$ with $|j| \geqq j_0$, it is clear that

$$(9) \qquad |X[\sigma^1(x, y; j^{-1})] - X[\sigma^1(x_0, y_0; j^{-1})]| \leqq 6\epsilon + \sum_{k=1}^{4} |X(\partial\tau_k)|$$

where τ_k, $k=1, \cdots, 4$, are right triangles with legs parallel to the x and y-axes with $|\tau_k|_2 \leqq \delta^2/2$ for $k=1, 2$ and $|\tau_k|_2 \leqq \delta N/2$ for $k=3, 4$. We conclude from the choice of δ that the left side of the inequality in (9) is majorized by 10ϵ. Consequently, $X[\sigma^1(x, y; j^{-1})]$ is continuous when restricted to S for $|j| \geqq j_0$ and $a(x, y)$ is a measurable function in S and therefore in R. Similarly, $b(x, y)$ is a measurable function in R. Consequently, it follows from (8) that both $a(x, y)$ and $b(x, y)$ are locally in L^1 on R.

Next, using the classical theory of additive functions of an interval [**3**, Chapter 3], it follows from (2) that there exists a function $c(x, y)$ which is locally in L^1 on R such that if τ is a 2-simplex in R as in (2) then

$$(10) \qquad X(\partial\tau) = \int_\tau c(x, y)dxdy.$$

Furthermore, it is clear from [**3**, p. 118] that

$$(11) \qquad |c(x, y)| \leqq H(x, y) \text{ almost everywhere in } R.$$

We shall designate the closed disc with center (x_0, y_0) and radius h by $D(x_0, y_0; h)$, and for $D(x_0, y_0; h) \subset R$, the integral $(\pi h^2)^{-1}\int_{D(x_0,y_0;h)} a(x, y)dxdy$ by $a^h(x_0, y_0)$. It follows from [**3**, Chapter 4], that there exists a measurable set $Q \subset R$ with $|R-Q|_2=0$ such that for every point (x, y) in Q, $a^h(x, y) \rightarrow a(x, y)$ and $b^h(x, y) \rightarrow b(x, y)$ as $h \rightarrow 0$.

If σ is a Q-good 1-simplex contained in R, we observe that for h small, $a^h(x, y)$ and $b^h(x, y)$ are continuous functions on σ. Consequently, it follows from the definition of Q and the fact that σ is Q-good that $a(x, y)$ and $b(x, y)$ are measurable functions on σ. Furthermore, it follows from (8) that

$$(12) \qquad \left|\int_\sigma |a(x, y)| \, dx\right| < \infty \quad \text{and} \quad \left|\int_\sigma |b(x, y)| \, dy\right| < \infty.$$

We observe that we have so far shown that the differential form $\omega = a(x, y)dx + b(x, y)dy$ meets conditions (3) and (4) in R. It follows from (1) and (7) that

$$(13) \qquad X(\sigma) = \int_\sigma \omega \qquad \text{for } \sigma \text{ in } R \text{ parallel to the } x \text{ or } y\text{-axes.}$$

We now show that

$$X(\sigma) = \int_\sigma \omega, \ \sigma \text{ a } Q\text{-good 1-simplex in } R. \tag{14}$$

By (12), we observe that the right side of (14) is well-defined. To establish (14), we suppose that σ is fixed and that there exist three concentric squares with sides parallel to the x and y-axes, S, S', and S'', with σ in S^0 and

$$S \subset S'^0 \subset S' \subset S''^0 \subset S'' \subset R.$$

We let $S(x, y; h)$ designate the square

$$S(x, y; h) = \{(u, v); \ x-h \leqq u \leqq x+h, \text{ and } y-h \leqq v \leqq y+h\},$$

and $\partial S(x, y; h)$ its positively oriented boundary. Then it follows immediately from (10) and (13) that

$$\int_{\partial S(x,y;h)} \omega = \int_{S(x,y;h)} c(x, y)dxdy \qquad \text{for } S(x, y; h) \subset S'', \tag{15}$$

and consequently that

$$\lim_{h\to 0} \int_{S'} \left| (4h^2)^{-1} \int_{\partial S(x,y;h)} \omega - c(x, y) \right| dxdy = 0. \tag{16}$$

Next, we choose a non-negative function $\lambda(x, y)$ which is in class C^∞ on E^2 and takes the value one in S and zero in $E^2 - S'$. We then set $a'(x, y) = \lambda(x, y)a(x, y)$, $b'(x, y) = \lambda(x, y)b(x, y)$, $c'(x, y) = \lambda(x, y)c(x, y) + b(x, y)\lambda_x(x, y) - a(x, y)\lambda_y(x, y)$ for (x, y) in S'' and $a'(x,y) = b'(x,y) = c'(x, y) = 0$ for (x, y) in $E^2 - S''$.

With these definitions, we now show that (16) implies that

$$\lim_{h\to 0} \int_{E^2} \left| (4h^2)^{-1} \int_{\partial S(x,y;h)} a'(u, v)du + b'(u, v)dv - c'(x, y) \right| dxdy = 0. \tag{17}$$

We first introduce an intermediate square S''' to S' and S''. Then there exists an h_0 such that for $0 < h < h_0$,

$$\begin{aligned} &\int_{E^2} \left| (4h^2)^{-1} \int_{\partial S(x,y;h)} a'(u, v)du + b'(u, v)dv - c'(x, y) \right| dxdy \\ &= \int_{S'''} \left| (4h^2)^{-1} \int_{\partial S(0,0;h)} a'(x+u, y+v)du + b'(x+u, y+v) \right. \\ &\qquad\qquad \left. - c'(x, y) \right| dxdy. \end{aligned} \tag{18}$$

Setting $\lambda(x+u, y+v) = \lambda(x, y) + u\lambda_x(x, y) + v\lambda_y(x, y) + \eta(x, y, u, v)$, we see that the right side of (18) is majorized by the following sum:

$$\int_{S'''} \lambda(x, y) \left| (4h^2)^{-1} \int_{\partial S(0,0;h)} a(x+u, y+v)du + b(x+u, y+v)dv \right.$$
$$\left. -c(x, y) \right| dxdy$$
$$+ \int_{S'''} \left| (4h^2)^{-1} \int_{\partial S(0,0;h)} \eta(x, y, u, v)\left[a(x+u, y+v)du \right.\right.$$
$$\left.\left. + b(x+u, y+v)dv\right] \right| dxdy$$
$$+ \int_{S'''} | \lambda_x(x, y) | \left| (4h^2)^{-1} \int_{\partial S(0,0;h)} ua(x+u, y+v)du \right| dxdy$$
$$+ \int_{S'''} | \lambda_y(x, y) | \left| (4h^2)^{-1} \int_{\partial S(0,0;h)} va(x+u, y+v)du + a(x, y) \right| dxdy$$
$$+ \int_{S'''} | \lambda_x(x, y) | \left| (4h^2)^{-1} \int_{\partial S(0,0;h)} ub(x+u, y+v)dv - b(x, y) \right| dxdy$$
$$+ \int_{S'''} | \lambda_y(x, y) | \left| (4h^2)^{-1} \int_{\partial S(0,0;h)} vb(x+u, y+v)dv \right| dxdy$$
$$= I_h^1 + I_h^2 + I_h^3 + I_h^4 + I_h^5 + I_h^6.$$

That $I_h^1 \to 0$ as $h \to 0$ follows immediately from (16) and the fact that $\lambda(x, y) = 0$ in $E^2 - S'$.

From Taylor's theorem for functions of several variables, it follows that there exists a constant K such that for (x, y) in S''' and all (u, v), $|\eta(x, y, u, v)| \leqq K(u^2+v^2)$. But then by Fubini's theorem for h small,

$$I_h^2 \leqq (4h^2)^{-1} 2K \left\{ \int_{-h}^{h} (u^2 + h^2) \left[\int_{S''} | a(x, y) | \, dxdy \right] du \right.$$
$$\left. + \int_{-h}^{h} (v^2 + h^2) \left[\int_{S''} | b(x, y) | \, dxdy \right] dv \right\}.$$

We conclude that $I_h^2 \to 0$ as $h \to 0$.

Observing that for h small and for almost every (x, y) in S'''

$$\int_{\partial S(0,0;h)} ua(x + u, y + v)du = \int_{\partial S(0,0;h)} u[a(x + u, y + v) - a(x, y)]du,$$

and letting K_1 be a constant such that $|\lambda_x(x, y)| \leqq K_1$ for (x, y) in E^2, we obtain that for h small

$$|I_h^3| \leqq K_1(4h^2)^{-1}\int_{S'''}\left\{\int_{-h}^{h}|u|\,[\,|a(x+u,y-h)-a(x,y)|\right.$$

$$\left.+\,|a(x+u,y+h)-a(x,y)|\,]du\right\}dxdy.$$

We conclude from Fubini's theorem that $I_h^3 \to 0$ as $h \to 0$. Next, we observe that for h small and almost every (x, y) in S''',

$$(4h^2)^{-1}\int_{\partial S(0,0;h)} va(x+u,y+v)du + a(x,y)$$

$$= (4h)^{2-1}\int_{\partial S(0,0;h)} v[a(x+u,y+v) - a(x,y)]du.$$

Consequently, we can use the same reasoning for I_h^4 as that used for I_h^3 and obtain that $I_h^4 \to 0$ as $h \to 0$.

Handling I_h^5 in the same manner as I_h^4 and I_h^6 in the same manner as I_h^3, we obtain $I_h^5 \to 0$ and $I_h^6 \to 0$ as $h \to 0$. (17) is therefore established.

Next, we let $\hat{a}'(x, y)$, $\hat{b}'(x, y)$, $\hat{c}'(x, y)$ represent respectively the Fourier transform of $a'(x, y)$, $b'(x, y)$, and $c'(x, y)$, i.e.,

$$\hat{a}'(x,y) = (4\pi^2)^{-1}\int_{E^2} a'(u,v)e^{-i(xu+yv)}dudv.$$

Observing that the Fourier transform of the function

$$\int_{\partial S(x,y;h)} a'(u,v)du + b'(u,v)dv$$

is

$$i[x\hat{b}'(x,y) - y\hat{a}'(x,y)]\int_{S(0,0;h)} e^{i(ux+yv)}dudv,$$

we conclude from (17) that

$$\hat{c}'(x,y) = i[x\hat{b}'(x,y) - y\hat{a}'(x,y)]. \tag{19}$$

We turn our attention once again to σ, S, and S', and to the establishing of (14). We can suppose that the squares S and S' are given by

$$S = \{(x,y); x_0 - \rho \leqq x \leqq x_0 + \rho, y_0 - \rho \leqq y \leqq y_0 + \rho\},$$
$$S' = \{(x,y); x_0 - \rho' \leqq x \leqq x_0 + \rho', y_0 - \rho' \leqq y \leqq y_0 + \rho'\}$$

with $0 < \rho < \rho'$.

From (13), we observe that (14) holds if σ is parallel to the x or

y-axes. Consequently we can assume that σ is a Q-good oriented line segment given by

$$y = \alpha x + \beta, \qquad x_1 \leqq x \leqq x_2,$$

where $\alpha \neq 0$ and $x_0 - \rho < x_1 < x_2 < x_0 + \rho$. We shall also assume that $\alpha > 0$, since a similar proof will establish (14) in case $\alpha < 0$.

Next, we define σ_h to be the oriented line segment $y = \alpha x + \beta$, $x_1 - h \leqq x \leqq x_2 - h$. Since X is a local L^1 1-cochain in R, (1) and (2) imply that $X(\sigma_h) \to X(\sigma)$ as $h \to 0$. Since σ is a Q-good line segment, (12) shows that

$$\int_{\sigma_h} \omega \to \int_{\sigma} \omega \quad \text{as} \quad h \to 0.$$

Furthermore $|S - (S \cap Q)|_2 = 0$. Consequently, by Fubini's theorem, with no loss in generality, we can assume from the start that the oriented line segments σ_1 and σ_2 are each Q-good where σ_1 and σ_2 are given as follows:

$$\sigma_1\colon\ x_2 \geqq x \geqq x_1, \qquad y = \alpha x_2 + \beta;$$
$$\sigma_2\colon\ x = x_1, \qquad \alpha x_2 + \beta \geqq y \geqq \alpha x_1 = \beta.$$

If we then designate by τ the oriented 2-simplex (oriented like E^2) having σ, σ_1, and σ_2 as its edges, we obtain that $\partial\tau = \sigma + \sigma_1 + \sigma_2$.

Next, we set for $t > 0$,

$$(20) \qquad \begin{aligned} a_t'(x, y) &= \int_{E^2} \hat{a}'(u, v) e^{-(u^2+v^2)t} e^{i(ux+vy)} du dv, \\ b_t'(x, y) &= \int_{E^2} \hat{b}'(u, v) e^{-(u^2+v^2)t} e^{i(ux+vy)} du dv, \\ c_t'(x, y) &= \int_{E^2} \hat{c}'(u, v) e^{-(u^2+v^2)t} e^{i(ux+vy)} du dv, \end{aligned}$$

and observe that $a_t'(x, y)$, $b_t'(x, y)$, and $c_t'(x, y)$ are in class C^∞ on E^2. Furthermore we obtain from (19) and (20) that

$$\frac{\partial b_t'(x, y)}{\partial x} - \frac{\partial a_t'(x, y)}{\partial y} = c_t'(x, y),$$

and consequently that for $t > 0$,

$$(21) \qquad \int_{\partial\tau} a_t'(x, y) dx + b_t'(x, y) dy = \int_{\tau} c_t'(x, y) dx dy.$$

Now, if we can show

$$
(22)\qquad
\begin{aligned}
&\int_{\sigma} a_t'(x, y)dx \to \int_{\sigma} a'(x, y)dx && \text{as } t \to 0,\\
&\int_{\sigma_1} a_t'(x, y)dx \to \int_{\sigma_1} a'(x, v)dx && \text{as } t \to 0,\\
&\int_{\sigma} b_t'(x, y)dy \to \int_{\sigma} b'(x, y)dy && \text{as } t \to 0,\\
&\int_{\sigma_2} b_t'(x, y)dy \to \int_{\sigma_2} b'(x, y)dy && \text{as } t \to 0,
\end{aligned}
$$

we shall have established (14). For the theory of multiple Fourier integrals (see [1, p. 22]) shows that $c_t'(x, y) \to c'(x, y)$ in the L^1-norm on E^2. Consequently, taking the limit on both sides of (21) as $t \to 0$, we obtain from (22) that

$$
(23)\qquad \int_{\partial\tau} a'(x, y)dx + b'(x, y)dy = \int_{\tau} c'(x, y)dxdy.
$$

But τ is contained in S, and in S, $a'(x, y) = a(x, y)$, $b'(x, y) = b(x, y)$ and $c'(x, y) = c(x, y)$. Therefore (23) implies that

$$
(24)\qquad \int_{\partial\tau} \omega = \int_{\tau} c(x, y)dxdy.
$$

Using (10) and the fact that $\partial\tau = \sigma + \sigma_1 + \sigma_2$, we infer from (24) that

$$
\int_{\sigma} \omega + \int_{\sigma_1} \omega + \int_{\sigma_2} \omega = X(\partial\tau) = X(\sigma) + X(\sigma_1) + X(\sigma_2).
$$

By (13), however, $X(\sigma_1) = \int_{\sigma_1}\omega$ and $X(\sigma_2) = \int_{\sigma_2}\omega$; consequently (14) will be established, once we show that (22) holds.

We establish (22) by showing that the first limit in (22) holds. A similar procedure will show that the other three limits in (22) hold.

In order to accomplish this, we designate the indicator function of the interval $x_0 - \rho' \leqq x \leqq x_0 + \rho'$ by $\lambda_1(x)$. Then since $\lambda(x, y) \leqq \lambda_1(x)$ for all (x, y) in E^2, we obtain from (8) on setting $g_1'(x) = g(x)\lambda_1(x)$ for $|x - x_0| \leqq \rho'$ and $g_1'(x) = 0$ for $|x - x_0| > \rho'$ that

$$
(25)\qquad |a'(x, y)| \leqq g_1'(x) \qquad \text{for } (x, y) \text{ in } E^2.
$$

Since $a^h(x, y) \to a(x, y)$ as $h \to 0$ for (x, y) in Q, we obtain from the theory of multiple Fourier integrals (see [2, Chapter 2]) that $a_t'(x, y)$

$\to a'(x, y)$ for (x, y) in S^0Q. Since σ is Q-good and contained in S^0, we have that

(26) $\lim_{t\to 0} a_t'(x, \alpha x+\beta)=a'(x, \alpha x+\beta)$ almost everywhere in $x_1 \leqq x \leqq x_2$.

By (12),

$$\int_{x_1}^{x_2} |\, a'(x, \alpha x + \beta)\,|\, dx < \infty. \tag{27}$$

From (20) and well-known facts concerning Fourier transforms we have that for $t>0$,

$$a_t'(x, y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} a'(x - u, y - v)\, \frac{e^{-|u|^2/4t}}{(4\pi t)^{1/2}}\, \frac{e^{-|v|^2/4t}}{(4\pi t)^{1/2}}\, dudv,$$

and consequently from (25) that for $t>0$

$$|\, a_t'(x, \alpha x + \beta)\,| \leqq (4\pi t)^{-1/2}\int_{-\infty}^{\infty} g_1'(x - u)e^{-|u|^2/4t}du. \tag{28}$$

Now $g_1'(x)$ is a non-negative function in L^1 on $-\infty<x<\infty$ which vanishes for $|x-x_0|>\rho'$. Consequently given $\epsilon>0$, there exists a $\delta_1>0$ such that if B is a measurable set on the real line with $|B|_1<\delta_1$, then

$$\int_B g_1'(x - u)dx < \epsilon \qquad \text{for all } u.$$

But then Fubini's theorem and (28) imply that for $t>0$,

$$\int_B |\, a_t'(x, \alpha x + \beta)\,|\, dx < \epsilon \qquad \text{for } |\, B\,|_1 < \delta_1. \tag{29}$$

Using Egoroff's theorem, (26), (27), and (29), we obtain

$$\lim_{t\to 0} \int_{x_1}^{x_2} a_t'(x, \alpha x + \beta)dx = \int_{x_1}^{x_2} a'(x, \alpha x + \beta)dx. \tag{30}$$

The first limit in (22) is therefore established. A similar procedure establishes the other three limits in (22), and as we have shown earlier, this establishes (14).

Summarizing, we have shown that the functions $a(x, y)$ and $b(x, y)$ are measurable in R, that there exist functions $g_1(x)$ and $g_2(y)$ which are locally in L^1 on R such that $|a(x, y)| \leqq g_1(x)$ and $|b(x, y)| \leqq g_2(y)$ for every (x, y) in R, and that there exists a measurable set $Q\subset R$

with $|R-Q|_2=0$ such that if σ is a Q-good 1-simplex in R, then $a(x, y)$ and $b(x, y)$ are measurable on σ and $X(\sigma)=\int_\sigma \omega$ where $\omega=a(x, y)dx+b(x, y)dy$.

In order to complete the proof that ω is a local L^1 differential 1-form in R, we need only show that if τ is a Q-excellent 2-simplex in R oriented like E^2 with two edges parallel to the x and y-axes, then $\left|\int_{\partial\tau}\omega\right| \leqq \int_\tau H(x, y)dxdy$ where $H(x, y)$ is the function in (2). However, this is immediate from (11). For since τ is Q-excellent,

$$\left|\int_{\partial\tau}\omega\right| = |X(\partial\tau)| = \left|\int_\tau c(x, y)dxdy\right| \leqq \int_\tau H(x, y)dxdy.$$

To complete the proof of part (a) of the theorem, we have to show that if $\omega_1(x, y)=a_1(x, y)dx+b_1(x, y)dy$ is a local L^1 differential 1-form and Q_1 is a measurable set in R satisfying (4) and (5) with respect to ω_1, and, furthermore, if

$$X(\sigma) = \int_\sigma \omega_1 \qquad \text{for every } Q_1\text{-good 1-simplex } \sigma \text{ in } R,$$

then

(31) $\quad a_1(x, y) = a(x, y)$ and $b_1(x, y) = b(x, y)$ almost everywhere in R.

To establish (31), set $Q_2=QQ_1$. Then $|R-Q_2|_2=0$. Next, let S be a square with sides parallel to the x and y-axes such that $S\subset R$. Then by hypothesis, if σ is a Q_2-good 1-simplex contained in S, $\int_\sigma\omega=\int_\sigma\omega_1$. Since both $a(x, y)$ and $a_1(x, y)$ are in L^1 on S, we consequently have from Fubini's theorem that

$$\int_{y_0-t}^{y_0+t}\int_{x_0-t}^{x_0+t}[a(x, y) - a_1(x, y)]dxdy = 0 \text{ if the square } S(x_0, y_0; t) \subset S.$$

But then Lebesgue's theorem [**3**, p. 118] tells us that $a(x, y)=a_1(x, y)$ almost everywhere in S and therefore almost everywhere in R. Similarly $b(x, y)=b_1(x, y)$ almost everywhere in R, (31) is established, and the proof of part (a) of the theorem is complete.

3. We now prove part (b) of the theorem. Let $\omega=a(x, y)dx+b(x, y)dy$ be the given local L^1 differential 1-form in R an let Q be the measurable set contained in R satisfying (4) and (5). By (3) and (4), $a(x, y)$ and $b(x, y)$ are each locally in L^1 on R. As before, letting $D(x_0, y_0; h)$ stand for the disc with center (x_0, y_0) and radius h, we set $a^h(x_0, y_0)=(\pi h^2)^{-1}\int_{D(x_0,y_0;h)}a(x, y)dxdy$ and similarly define $b^h(x_0, y_0)$. Now, $a^h(x, y)\to a(x, y)$ and $b^h(x, y)\to b(x, y)$ almost everywhere in R as $h\to 0$. We set

$$(32)\qquad Q' = \{(x, y); (x, y) \text{ in } Q, a^h(x, y) \to a(x, y) \text{ and } b^h(x, y) \to b(x, y) \text{ as } h \to 0\},$$

and note that both $|Q-Q'|_2=0$ and $|R-Q'|_2=0$.

We next establish the following fact: if τ is a Q'-excellent 2-simplex in R oriented like E^2 and if $H(x, y)$ is the function given in (5) which is locally in L^1 on R, then

$$(33)\qquad \left|\int_{\partial\tau} \omega\right| \leqq \int_\tau H(x, y)dxdy.$$

It is to be noticed that in (33), τ is an arbitrary Q'-excellent 2-simplex, i.e., we are not restricting it to be a right triangle.

In establishing (33), we first notice that if follows from (5) that if $S(x, y; h)$ (i.e., the square oriented like E^2 with center (x, y) and side $2h$) is a Q'-excellent square contained in R, then

$$(34)\qquad \left|\int_{\partial S[x,y;h]} \omega\right| \leqq \int_{S[x,y;h]} H(x, y)dxdy.$$

Next, we introduce four bounded domains D_1, D_2, D_3 and D_4 with $\tau\subset D_1\subset\overline{D}_1\subset D_2\subset\overline{D}_2\subset D_3\subset\overline{D}_3\subset D_4\subset\overline{D}_4\subset R$ and a non-negative function $\lambda(x, y)$ in class C^∞ on E^2 which takes the value one in $\overline{D}_1$ and the value zero in $E^2-\overline{D}_2$. We define $a'(x, y)=\lambda(x, y)a(x, y)$ and $b'(x, y)=\lambda(x, y)b(x, y)$ where $a'(x, y)$ and $b'(x, y)$ are set equal to zero in E^2-R. Then we set for $t>0$,

$$a_t'(x, y) = (4\pi t)^{-1}\int_{E^2} a'(x-u, y-v)e^{-(u^2+v^2)/4t}dudv,$$

$$b_t'(x, y) = (4\pi t)^{-1}\int_{E^2} b'(x-u, y-v)e^{-(u^2+v^2)/4t}dudv,$$

$$\omega_t'(x, y) = a_t'(x, y)dx + b_t'(x, y)dy.$$

Since τ is Q'-excellent, we see from the definition of Q' and well-known facts concerning the summability of multiple Fourier integrals [**2**, Chapter 2] that $a_t'(x, y)\to a'(x, y)$ and $b_t'(x, y)\to b'(x, y)$ as $t\to 0$ almost everywhere in the 1-dimensional sense on $\partial\tau$. We consequently infer from (4), using the same technique employed in establishing (22), that

$$(35)\qquad \int_{\partial\tau}\omega_t'(x, y) \to \int_{\partial\tau}\omega'(x, y) \qquad \text{as } t\to 0.$$

Next, setting $\lambda(x, y)H(x, y) = \lambda_y(x, y)a(x, y) = \lambda_x(x, y)b(x, y) = 0$ in $E^2 - R$, we shall establish that for (x, y) in D_1,

$$\left| \frac{\partial}{\partial x} b_t'(x, y) - \frac{\partial}{\partial y} a_t'(x, y) \right|$$

$$\tag{36} \begin{aligned} \leqq (4\pi t)^{-1} \int_{E^2} e^{-(u^2+v^2)/4t} \{ &\lambda(x - u, y - v)H(x - u, y - v) \\ &+ | \lambda_u(x - u, y - v)b(x - u, y - v) \\ &- \lambda_v(x - u, y - v)a(x - u, y - v) | \} dudv. \end{aligned}$$

We observe that once (36) is established, (33) follows. For the right side of (36) tends in L^1-norm on E^2 to $\lambda(x, y)H(x, y) - |\lambda_x(x, y)b(x, y) - \lambda_y(x, y)a(x, y)|$ as $t \to 0$ and this function is equal to $H(x, y)$ for (x, y) in τ. We consequently conclude from (36) that

$$\tag{37} \limsup_{t \to 0} \int_\tau \left| \frac{\partial}{\partial x} b_t'(x, y) - \frac{\partial}{\partial y} a_t'(x, y) \right| dxdy \leqq \int_\tau H(x, y) dxdy.$$

However, since $\omega_t'(x, y)$ is a differential form in class C^∞, we conclude from Stoke's theorem and (37) that

$$\tag{38} \lim_{t \to 0} \left| \int_{\partial \tau} \omega_t'(x, y) \right| \leqq \int_\tau H(x, y) dxdy.$$

But then using (35) and the fact that $\omega'(x, y) = \omega(x, y)$ in a domain containing τ, we obtain (33) from (38).

We now proceed with the establishing of (36). In the ensuing discussion, we shall set for $t > 0$,

$$N_t(u, v) = e^{-(u^2+v^2)/4t}/(4\pi t)^{-1}.$$

Fix (x_0, y_0) in D_1 and let h_j equal the distance between $\overline{D}_j$ and the boundary of D_{j+1}, $j = 1, 2, 3$, and set $h_0 = \min(h_1, h_2, h_3)/4$.

Then for $0 < h < h_0$,

$$\begin{aligned} \int_{\partial S[x_0, y_0; h]} \omega_t'(x, y) &= \int_{E^2} N_t(u, v) \left[\int_{\partial S[x_0, y_0; h]} \omega'(x - u, y - v) \right] dudv \\ &= I_{1,h} + I_{2,h} \end{aligned}$$

where

$$I_{1,h} = \int_{E^2} N_t(u, v) \lambda(x_0 - u, y_0 - v) \left[\int_{\partial S[x_0, y_0; h]} \omega(x - u, y - v) \right] dudv$$

and

$$I_{2,h} = \int_{E^2} N_t(u, v) \left\{ \int_{\partial S[x_0, y_0; h]} [\lambda(x - u, y - v) - \lambda(x_0 - u, y_0 - v)] \right.$$
$$\left. \cdot \omega(x - u, y - v) \right\} dudv.$$

It is clear from Stokes' theorem and the fact that ω_t' is a differential form in class C^∞ that

$$\left| \frac{\partial b_t'(x_0, y_0)}{\partial x} - \frac{\partial a_t'(x_0, y_0)}{\partial y} \right| \leqq \limsup_{h \to 0} \frac{|I_{1,h}|}{4h^2} + \limsup_{h \to 0} \frac{|I_{2,h}|}{4h^2} \cdot$$

Therefore to establish (36), we need only show that

$$\limsup_{h \to 0} |I_{1,h}| / 4h^2$$
$$\leqq \int_{E^2} N_t(u, v) \lambda(x_0 - u, y_0 - v) H(x_0 - u, y_0 - v) dudv \tag{39}$$

and that

$$\limsup_{h \to 0} |I_{2,h}| / 4h^2 \leqq \int_{E^2} N_t(u, v) \,|\, \lambda_u(x_0 - u, y_0 - v) b(x_0 - u, y_0 - v)$$
$$- \lambda_v(x_0 - u, y_0 - v) a(x_0 - u, y_0 - v) \,|\, dudv. \tag{40}$$

We observe that

$$I_{1,h} = \int_{E^2} N_t(u - x_0, v - y_0) \lambda(u, v) \left[\int_{\partial S(u,v;h)} \omega(x, y) \right] dudv$$
$$= \int_{D_2} N_t(u - x_0, v - y_0) \lambda(u, v) \left[\int_{\partial S(u,v;h)} \omega(x, y) \right] dudv.$$

But for almost every (u, v) in D_2, $S(u, v; h)$ is Q'-excellent. Consequently, from (34) we obtain that

$$|I_{1,h}| \leqq \int_{D_2} N_t(u - x_0, v - y_0) \lambda(u, v)$$
$$\cdot \left[\int_{S(0,0;h)} H(x + u, y + u) dxdy \right] dudv. \tag{41}$$

But then (39) follows immediately from (41) on observing that

$(4h^2)^{-1} \int_{S(0,0;h)} H(x+u, y+v) dxdy \to H(u, v)$ as $h \to 0$ in the L^1-norm on D_2.

To establish (40), we observe that

$$
(42)\qquad \begin{aligned} I_{2,h} = \int_{D_3} & N_t(u-x_0, v-y_0) \\ & \cdot \left\{ \int_{\partial S(0,0;h)} [\lambda(x+u, y+v) - \lambda(u, v)]\omega(x+u, y+v) \right\} dudv. \end{aligned}
$$

Next, we set

$$
\lambda(x+u, y+v) - \lambda(u, v) = \lambda_u(u, v)x + \lambda_v(u, v)y + \eta(x, y, u, v)
$$

and obtain that there is a constant K such that

$$
|\eta(x, y, u, v)| \leqq K(x^2 + y^2) \qquad \text{for all } (x, y) \text{ and } (u, v).
$$

Observing that

$$
\begin{aligned} \int_{D_3} N_t(u - x_0, v - y_0) \left[\int_{-h}^{h} (x^2 + h^2) \, | a(x + u, h + v) | \, dx \right] dudv & \\ = o(h^2) \ \text{ as } h \to 0, & \end{aligned}
$$

we conclude from (42) that

$$
(43)\qquad \begin{aligned} I_{2,h} = \int_{D_3} & N_t(u - x_0, v - y_0) \\ & \cdot \left\{ \int_{\partial S(0,0;h)} [\lambda_u(u, v)x + \lambda_v(u, v)y]\omega(x + u, y + v) \right\} dudv \\ & + o(h^2). \end{aligned}
$$

We next observe that

$$
\begin{aligned} \int_{D_3} & N_t(u - x_0, v - y_0)\lambda_u(u, v) \\ & \cdot \left\{ \int_{\partial S(0,0;h)} x[a(x + u, y + v) - a(u, v)]dx \right\} dudv = o(h^2) \end{aligned}
$$

and conclude from (43) that

$$
I_{2,h} = I_{3,h} + o(h^2) \quad \text{where}
$$

$$
(44)\qquad \begin{aligned} I_{3,h} = \int_{D_3} N_t(u - x_0, v - y_0) \left\{ \int_{\partial S(0,0;h)} \right. & \lambda_v(u, v)ya(x + u, y + v)dx \\ & \left. + \lambda_u(u, v)xb(x + u, y + v)dy \right\} dudv. \end{aligned}
$$

Setting

$$(45)\quad M = \int_{D_3} N_t(u - x_0, v - y_0)[-\lambda_v(u, v)a(u, v) + \lambda_u(u, v)b(u, v)]dudv,$$

we observe that

$$(4h^2)^{-1}I_{3,h} - M = (4h^2)^{-1}\int_{D_3} N_t(u - x_0, v - y_0) \cdot \left\{\int_{\partial S(0,0;h)} \lambda_v(u, v)y[a(x + u, y + v) - a(u, v)]dx + \lambda_u(u, v)x[b(x + u, y + v) - b(u, v)]dy\right\} dudv.$$

We conclude from Fubini's theorem that

$$(46)\qquad \lim_{h\to 0} (4h^2)^{-1}I_{3,h} = M.$$

But (44), (45), and (46) give us (40). Consequently, (36) and therefore (33) is established.

We now proceed with the proof of part (b) of the theorem.

Letting $\sigma_n = (p_n, q_n)$ designate the line segment oriented from p_n to q_n, we shall say $\{\sigma_n\}_{n=1}^\infty$ is a Cauchy sequence in R if there exists a convex, compact subset C of R such that σ_n lies in C for each n and if $\{p_n\}_{n=1}^\infty$ and $\{q_n\}_{n=1}^\infty$ are Cauchy sequences with respect to the usual metric in E^2. We say that the sequence $\{\sigma_n\}_{n=1}^\infty$ tends to the 1-simplex $\sigma_0 = (p_0, q_0)$ and write $\sigma_n \to \sigma_0$ as $n \to \infty$ if $p_n \to p_0$ and $q_n \to q_0$. It is now clear that if $\{\sigma_n\}_{n=1}^\infty$ is a Cauchy sequence in R, there exists a 1-simplex σ_0 such that $\sigma_n \to \sigma_0$ as $n \to \infty$.

From (4) and the definition of Q', it follows that if σ is a Q'-good 1-simplex in R, then $\int_\sigma |a(x, y)| dx < \infty$ and $\int_\sigma |b(x, y)| dy < \infty$. Consequently, if σ is a Q'-good 1-simplex in R, we define $X(\sigma) = \int_\sigma \omega$ and establish the following fact:

(47) if $\{\sigma_n\}_{n=1}^\infty$ is a Cauchy sequence of Q'-good 1-simplices in R, then $\{X(\sigma_n)\}_{n=1}^\infty$ is a Cauchy sequence of real numbers.

Since there exists $\sigma_0 = (p_0, q_0)$ in R such that $\sigma_n \to \sigma_0$ as $n \to \infty$, we can suppose from the start in establishing (47) that there exists a compact convex $C \subset R$ such that σ_n is in C^0 for $n = 0, 1, 2, \cdots$.

Next, let $x_1 \leqq x \leqq x_2$ and $y_1 \leqq y \leqq y_2$ be the smallest rectangle with sides parallel to the x and y-axes containing C. Then with $g_1(x)$ and $g_2(y)$ given by (4), we have that

$$\int_{x_1}^{x_2} g_1(x)dx < \infty \quad \text{and} \quad \int_{y_1}^{y_2} g_2(y)dy < \infty.$$

Set $K=[(x_2-x_1)^2+(y_2-y_1)^2]^{1/2}$ and let $\epsilon>0$ be given. Choose $\delta>0$ such that if $|x_3-x_4|\leqq\delta$ and if $|y_3-y_4|\leqq\delta$ with $x_1\leqq x_3\leqq x_4\leqq x_2$ and $y_1\leqq y_3\leqq y_4\leqq y_2$, then

$$\int_{x_3}^{x_4} g_1(x)dx < \epsilon \quad \text{and} \quad \int_{y_3}^{y_4} g_2(y)dy < \epsilon, \tag{48}$$

and furthermore such that if τ is a 2-simplex contained in C and oriented like E^2 with $|\tau|_2\leqq\delta$ and if $H(x, y)$ is given by (5), then

$$\int_\tau H(x, y)dxdy < \epsilon. \tag{49}$$

Next, choose n_0 sufficiently large so that for $n\geqq n_0$,

$$|p_n - p_0| \leqq \min\,[\delta/4(K+1),\ |\sigma_0|_1/8] \text{ and}$$
$$|q_n - q_0| \leqq \min\,[\delta/4(K+1),\ |\sigma_0|_1/8].$$

We propose to show that

$$|X(\sigma_m) - X(\sigma_n)| \leqq 14\epsilon \qquad \text{for } m \text{ and } n \geqq n_0, \tag{50}$$

which fact will establish (47).

First, suppose that the 1-simplices (p_m, p_n), (q_n, q_m), and (p_m, q_n) are also Q'-good. (Notice that because of convexity, these last three simplices also lie in C.) Then

$$\begin{aligned} |X(\sigma_m) - X(\sigma_n)| & \\ \leqq & \ |X(p_m, q_m) + X(q_m, q_n) + X(q_n, p_m)| \\ & + |X(p_m, q_n) + X(q_n, p_n) + X(p_n, p_m)| \\ & + |X(q_n, q_m)| + |X(p_m, p_n)|. \end{aligned}$$

Observing that the Q'-excellent oriented 2-simplex whose oriented boundary is given by $(p_m, q_m)+(q_m, q_n)+(q_n, p_m)$ has an area bounded by $K\delta/4(K+1)$ with a similar remark holding for the other 2-simplex whose boundary is given by $(p_m, q_n)+(q_n, p_n)+(p_n, p_m)$, we conclude from (33), (49) and (48) that $|X(\sigma_m)-X(\sigma_n)|<6\epsilon$ if m and $n\geqq n_0$. (50) is therefore established in this case.

Next, suppose that at least one of the 1-simplices (p_m, p_n), (q_n, q_m), (p_m, q_n) is not Q'-good. Since σ_m and σ_n are each Q'-good, it follows from the choice of n_0 that σ_m and σ_n are not collinear; consequently, we can replace p_n by p_n' and q_n by q_n' where p_n' and q_n' each lie on σ_n and neither lie on the line determined by σ_m and where $|p_n-p_n'|$ and $|q_n-q_n'|$ are both less than $\min[\delta/4(K+1),\ |\sigma_0|_1/8]$. Then with $\sigma_n'=(p_n', q_n')$, we have that σ_n' is Q'-good and furthermore that

$$(51) \qquad |X(\sigma_n') - X(\sigma_n)| < 4\epsilon.$$

Next, using Fubini's theorem, choose p_m' and q_m' on σ_m such that (p_n', p_m'), (q_n', p_m'), and (q_n', q_m') are each Q'-good and such that both $|p_m - p_m'|$ and $|q_m - q_m'|$ are less than $\min[\delta/4(K+1), |\sigma_n|_1/8]$. Then with $\sigma_m' = (p_m', q_m')$, we have that σ_m' is Q'-good and furthermore that

$$(52) \qquad |X(\sigma_m') - X(\sigma_m)| < 4\epsilon.$$

But now from the first case considered, $|X(\sigma_m') - X(\sigma_n')| < 6\epsilon$. This fact combined with (51) and (52) gives us (50), and consequently (47).

If σ is a Q'-good simplex in R and if $\{\sigma_n\}_{n=1}^{\infty}$ is a Cauchy sequence of Q'-good 1-simplices in R with $\sigma_n \to \sigma$ as $n \to \infty$, it is clear from (47) that $X(\sigma_n) \to X(\sigma)$ as $n \to \infty$. On the other hand, if σ is a 1-simplex in R which is not Q'-good, we can use (47) to define $X(\sigma)$. In particular, if σ is such a 1-simplex, we can by Fubini's theorem always find a Cauchy sequence of Q'-good 1-simplices $\{\sigma_n\}_{n=1}^{\infty}$ in R such that $\sigma_n \to \sigma$. By (47), $\lim_{n\to\infty} X(\sigma_n)$ exists. We define $X(\sigma) = \lim_{n\to\infty} X(\sigma_n)$. It is clear from (47) that $X(\sigma)$ is well defined, that $X(-\sigma) = -X(\sigma)$, and furthermore that if $\sigma = \sigma_1 + \cdots + \sigma_n$ where $\sigma, \sigma_1, \cdots, \sigma_n$ are collinear and similarly oriented, that $X(\sigma) = X(\sigma_1) + \cdots + X(\sigma_n)$. Consequently, X is a 1-cochain in R.

To show that X is a local L^1 1-cochain in R, we first observe that if σ is a Q'-good 1-simplex in R parallel to and oriented like the x-axis, then

$$|X(\sigma)| = \left| \int_\sigma a(x, y)dx \right| \leqq \int_\sigma g_1(x)dx.$$

If σ is a 1-simplex in R parallel to and oriented like the x-axis but not Q'-good, then, using Fubini's theorem, we select a Cauchy sequence $\{\sigma_n\}_{n=1}^{\infty}$ of Q'-good 1-simplices in R which are parallel and oriented like the x-axis and furthermore such that $\sigma_n \to \sigma$ as $n \to \infty$. Then $X(\sigma_n) \to X(\sigma)$ and $\int_{\sigma_n} g_1(x)dx \to \int_\sigma g_1(x)dx$. Consequently, $|X(\sigma)| \leqq \int_\sigma g_1(x)dx$. We conclude that (1) holds for X.

If τ is a Q'-excellent 2-simplex in R oriented like E^2, then by (33),

$$(53) \qquad |X(\partial\tau)| = \left| \int_{\partial\tau} \omega \right| \leqq \int_\tau H(x, y)dxdy.$$

If τ is a 2-simplex in R oriented like E^2 but not Q'-excellent and $\partial\tau = (p, q) + (q, r) + (r, p)$, we select, using Fubini's theorem, a sequence of Q'-excellent 2-simplices, $\{\tau_n\}_{n=1}^{\infty}$, with $\tau_n \subset \tau$ and each oriented like E^2 and such that $\partial\tau_n=(p_n, q_n)+(q_n, r_n)+(r_n, p_n)$ with $p_n\to p$, $q_n\to q$, and $r_n\to r$ as $n\to\infty$. Then from the definition of X, $X(\partial\tau_n)\to X(\partial\tau)$. Furthermore, since $H(x, y)$ is locally in L^1 on R, $\int_{\tau_n}H(x, y)dxdy\to\int_\tau H(x, y)dxdy$. We conclude from (53) that $|X(\partial\tau)| \leqq \int_\tau H(x, y)dxdy$. Therefore (2) holds, and we have shown that X is a local L^1 1-cochain in R.

We next have to show that $X(\sigma)=\int_\sigma\omega$ if σ is a Q-good 1-simplex in R. First suppose σ is a Q-good 1-simplex in R parallel to the x-axis, say $\sigma=(p, q)$ where $p=(x_0, y_0)$ and $q=(x_1, y_0)$. With no loss in generality, we can assume by Fubini's theorem, (1), and (4), that there exists an h_0 such that the two 1-simplices parallel to the y-axis whose ordered end points are given by (x_0, y_0), (x_0, y_0+h_0), and (x_1, y_0), (x_1, y_0+h_0) are each Q'-good. Call these 1-simplices σ' and σ'', respectively. Then it follows that there exists a Cauchy sequence of Q'-good 1-simplices in R, $\{\sigma_n\}_{n=1}^\infty$, with the following properties: $\sigma_n\to\sigma$ as $n\to\infty$, and the end points of each σ_n lie on σ' and σ''. By definition, $X(\sigma)=\lim_{n\to\infty} X(\sigma_n)$. On the other hand, it follows from (4) and (5) that $\lim_{n\to\infty}\int_{\sigma_n}\omega=\int_\sigma\omega$. Also, by definition $X(\sigma_n)=\int_{\sigma_n}\omega$. We conclude that $X(\sigma)=\int_\sigma\omega$.

Similarly, if σ is a Q-good 1-simplex in R parallel to the y-axis, $X(\sigma)=\int_\sigma\omega$.

Next, let σ be a Q-good 1-simplex in R which is not Q'-good and which is not parallel to either the x or y-axes. In particular, let σ be the oriented 1-simplex: $y=\alpha x+\beta$, $x_1\leqq x\leqq x_2$ where $\alpha>0$. (A similar argument will prevail in the case $\alpha<0$.) With no loss in generality, we also can suppose that the rectangle $\{(x, y); x_1\leqq x\leqq x_2 \text{ and } \alpha x_1+\beta\leqq y \leqq\alpha x_2+\beta\}$ is contained in R and furthermore that the two 1-simplices whose ordered end points are $(x_1, \alpha x_1+\beta)$, $(x_2, \alpha x_1+\beta)$ and $(x_2, \alpha x_1+\beta)$, $(x_2, \alpha x_2+\beta)$ are each Q'-good.

Next, using Fubini's theorem, we select a double sequence of points $\{x_k^n\}$, $k=0, \cdots, n$ and $n=1, 2, \cdots$ with the following properties:

(i) $x_1=x_0^n<x_1^n<\cdots<x_n^n=x_2$;

(ii) the 1-simplices σ_k^n and σ'^n_{k+1} whose ordered end points are respectively $(x_k^n, \alpha x_k^n+\beta)$, $(x_{k+1}^n, \alpha x_k^n+\beta)$ and $(x_{k+1}^n, \alpha x_k^n+\beta)$ and $(x_{k+1}^n, \alpha x_{k+1}^n+\beta)$ are each Q'-good;

(iii) the 2-simplices τ_k^n oriented like E^2 and determined by σ_k^n and σ'^n_{k+1} are such that $\lim_{n\to\infty}\sum_{k=0}^{n-1}|\tau_k^n|_2=0$.

We observe from properties (i), (ii), and (iii) that

$$\sum_{k=0}^{n-1} X(\sigma_k^n) + \sum_{k=1}^{n} X(\sigma_k'^n) - X(\sigma) = \sum_{k=0}^{n-1} X(\partial\tau_k^n). \tag{54}$$

However, X is a local L^1-cochain in R. Therefore

$$\left| \sum_{k=0}^{n-1} X(\partial \tau_k^n) \right| \leqq \sum_{k=0}^{n-1} \int_{\tau_k^n} H(x, y) dxdy,$$

and we conclude from property (iii) and (54) that

$$X(\sigma) = \lim_{n \to \infty} \left[\sum_{k=0}^{n-1} X(\sigma_k^n) + \sum_{k=1}^{n} X(\sigma_k'^n) \right]. \tag{55}$$

Similarly,

$$\sum_{k=0}^{n-1} \int_{\sigma_k^n} \omega + \sum_{k=1}^{n} \int_{\sigma_k'^n} \omega - \int_\sigma \omega = \sum_{k=0}^{n-1} \int_{\partial \tau_k^n} \omega,$$

and by (5)

$$\left| \int_{\partial \tau_k^n} \omega \right| \leqq \int_{\tau_k^n} H(x, y) dxdy.$$

We conclude, as before, that

$$\int_\sigma \omega = \lim_{n \to \infty} \left[\sum_{k=0}^{n-1} \int_{\sigma_k^n} \omega + \sum_{k=1}^{n} \int_{\sigma_k'^n} \omega \right]. \tag{56}$$

But by property (ii) and the definition of X,

$$X(\sigma_k^n) = \int_{\sigma_k^n} \omega \quad \text{and} \quad X(\sigma_k'^n) = \int_{\sigma_k'^n} \omega.$$

We therefore obtain from (55) and (56) that $X(\sigma) = \int_\sigma \omega$. We conclude that $X(\sigma) = \int_\sigma \omega$ for every Q-good 1-simplex σ in R.

To complete the proof of part (b) of the theorem we have to show that if ω_1 is a local L^1 differential 1-form in R which is equivalent to ω and if ω_1 gives rise to the local L^1 1-cochain in R designated by X_1, then

$$X(\sigma) = X_1(\sigma) \qquad \text{for every 1-simplex } \sigma \text{ in } R. \tag{57}$$

To establish (57), let $\omega_1(x, y) = a_1(x, y)dx + b_1(x, y)dy$. Let Q_1 and Q_1' play the analogous roles for ω_1 that Q and Q' play for ω. Let $Q_2 = Q'Q_1'$. Then, $|R - Q_2|_2 = 0$. Furthermore, since $a(x, y) = a_1(x, y)$ and $b(x, y) = b_1(x, y)$ almost everywhere in R, we have that for (x_0, y_0) in R and $h > 0$ and small, $a^h(x_0, y_0) = a_1^h(x_0, y_0)$ and $b^h(x_0, y_0) = b_1^h(x_0, y_0)$. We conclude from the definition of Q_2, Q', and Q_1', that $a(x, y) = a_1(x, y)$ and $b(x, y) = b_1(x, y)$ for (x, y) in Q_2. But if σ is a Q_2-good 1-simplex in R, $X(\sigma) = \int_\sigma \omega$, $X_1(\sigma) = \int_\sigma \omega_1$, and $\int_\sigma \omega = \int_\sigma \omega_1$. Consequently,

$$X(\sigma) = X_1(\sigma) \qquad \text{for every } Q_2\text{-good 1-simplex } \sigma \text{ in } R. \tag{58}$$

If σ is an arbitrary 1-simplex in R, there exists a Cauchy sequence of Q_2-good 1-simplices in R, $\{\sigma_n\}_{n=1}^{\infty}$, such that $\sigma_n \to \sigma$ as $n \to \infty$. But then $X(\sigma) = \lim_{n\to\infty} X(\sigma_n)$ and $X_1(\sigma) = \lim_{n\to\infty} X_1(\sigma_n)$, and (57) follows immediately from (58). The proof of part (b) of the theorem is complete.

We conclude with the remark that the methods and techniques used here can be employed to obtain similar results for r-cochains in E^n.

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University of Oregon