THE EQUIVALENCE, FOR VARIETIES OF SEMIGROUPS, OF TWO PROPERTIES CONCERNING CONGRUENCE RELATIONS

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An equivalence relation σ on a semigroup S is called a congruence if $a\ \sigma\ b$ and $c\ \sigma\ d$ imply $(ac)\sigma\ (bd)$, for all $a, b, c, d \in S$. There is a fairly obvious correspondence between congruences on S and homomorphic images of S. For notational convenience we shall denote by S' the semigroup S with an identity element (if one is not already present) adjoined.

If $a \in S$, there is at least one congruence (viz., the identity relation, having a single element in each of its equivalence classes) for which $\{a\}$ is an equivalence class. Teissier [5] has essentially shown that this congruence is the only one having $\{a\}$ as a class if and only if:

(1) For each $b, c \in S$ with $b \neq c$, there exist $x, y \in S'$ such that exactly one of the pair xby, xcy equals a.

We shall call S *disjunctive* if (1) holds for all $a \in S$. Thus we may say that a disjunctive semigroup is characterized by the property that the identity relation is uniquely determined by each of its classes.

Now suppose σ and ρ are congruences on S. We define a relation $\sigma \circ \rho$, called the product of σ and ρ, by: $a\ (\sigma \circ \rho)\ b$ if and only if $a\ \sigma\ x$ and $x\ \rho\ b$ for some $x \in S$. The assumption that every pair σ, ρ of congruences on S is *permutable* (in the sense that $\sigma \circ \rho = \rho \circ \sigma$) has a number of interesting consequences, e.g., an analogue of the Jordan-Hölder Theorem, in which one speaks of chains of congruences in place of chains of subgroups. See Birkhoff [1], Chapter VI (especially Theorem 5, page 87), where further references are given.

It is easy to see that the two conditions which we have considered (viz., disjunctivity and congruence-permutability) are not equivalent. For let $S = \{1, 2, \cdots, n\}$, where $n \geqq 3$, with the semigroup operation given by: $x \circ y = x + y$ if $x + y \leqq n$, $x \circ y = n$ if $x + y > n$. Then S is congruence-permutable, but not disjunctive.

A family V of semigroups is called a *variety* of semigroups if V contains all subsemigroups, all homomorphic images, and all direct prod-

[1] The results reported here were contained in the author's dissertation (Tulane University, 1960) written under the direction of Professor A. H. Clifford.

ucts of elements of V. Malcev [3] studied varieties of general algebraic systems, and raised the question whether, for a variety, disjunctivity is equivalent to congruence-permutability. Thurston [6] has shown that, for general algebraic systems, the answer is no. The purpose of this note is to announce the following:

THEOREM. *Let V be a variety of semigroups. Then each $S\in V$ is disjunctive if and only if each $S\in V$ is congruence-permutable.*

To prove this, one first shows that a disjunctive semigroup of more than two elements either is simple (i.e., contains no ideals except S itself) or is 0-simple (i.e., contains a zero element 0, and no ideals except S and $\{0\}$), and then that a semigroup belonging to a disjunctive variety must be a periodic group. Next, it is easy to see that in a congruence-permutable semigroup the ideals form a chain under inclusion. Then one shows that a semigroup belonging to congruence-permutable variety must be simple, and finally must be a periodic group. Thus, one can conclude that for a variety V of semigroups the following are equivalent: each $S\in V$ is disjunctive, each $S\in V$ is congruence-permutable, each $S\in V$ is a group, each $S\in V$ is a periodic group.

At several points in the proof known results of semigroup theory (chiefly from Rees [4] and Clifford [2]) are used. Also, some steps can be shortened by using theorems of Malcev [3] and Thurston [6] concerning varieties of general algebraic systems.

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