ON THE GROUP $\mathcal{E}[X]$ OF HOMOTOPY EQUIVALENCE MAPS

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Let X be a CW-complex; we shall consider the group[2]

$$\mathcal{E}[X]$$

formed by the homotopy classes of equivalence maps from X into itself with the operation induced by the composition of maps. It is clear to see that this group depends only on the homotopy type of X, hence should be determined by the known homotopy invariants of X. This is the problem which we shall try to study here. In fact, there exists a spectral sequence converging to $\mathcal{E}[X]$, whose initial terms are given, roughly speaking, by the cohomology of X and the automorphism group of its homotopy group.

Besides the satisfaction of curiosity, the group $\mathcal{E}[X]$ seems to have other interests. For example, it operates canonically on the special cohomology group [1] of X

$$\mathcal{E}[X] \times K(X) \to K(X)$$

and the quotient will be smaller than $K(X)$. In fact we can determine, more generally, the quotient

$$[X, G]/\mathcal{E}[X]$$

of the operation

$$\mathcal{E}[X] \times [X, G] \to [X, G]$$

where $[X, G]$ denotes the group of homotopy classes of maps from X into a topological group G. This may be considered as a first approach to determine the orbit space of the operation

$$(\mathcal{E}[X] \times \mathcal{E}[Y]) \times [X, Y] \to [X, Y]$$

where $[X, Y]$ is the set of homotopy classes of maps from X into Y.

The study of the group $\mathcal{E}[X]$ gives also some information about the image and the kernel of the canonical homomorphism (inversely, they determine the group $\mathcal{E}[X]$)

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[2] This group is also studied by M. Arkowitz, M. G. Barratt, C. R. Curjel, D. W. Kahn and P. Olum in the special case.

$$f\colon \mathcal{E}[X] \to \prod_{p\geqq 1} \operatorname{Aut} \pi_p(X)$$

(where

$$\operatorname{Aut} \pi_p(X)$$

denotes the group of automorphisms of the pth homotopy group of X), by the differentials of the spectral sequence. This can be considered as a weak form of Dehn's lemma [6] in higher dimension.

Finally, the determination of the group $\mathcal{E}[X]$ may reduce the study of the isotopy group of all diffeomorphisms [4] of a manifold X to the study of that invariant subgroup which consists only of those isotopy classes of diffeomorphisms which are homotopic to the identity map of X.

We give here only the spectral sequence and illustrate it by the particular case when X has only two nonzero homotopy groups. Details will appear elsewhere.

Let π be an abelian group and X a space; then there is a natural way for the group Aut π to operate on the cohomology group of the space X with coefficients in π, which will be denoted by

$$\theta\colon H^*(X, \pi) \times \operatorname{Aut} \pi \to H^*(X, \pi).$$

And, if ξ is a fixed element of $H^*(X, \pi)$, we shall use the notation

$$\operatorname{Aut}_\xi \pi$$

to denote the subgroup of stablizers of ξ by θ; that is, those elements α of Aut π which verify the relation

$$\theta(\xi, \alpha) = \xi \Leftrightarrow \alpha \in \operatorname{Aut}_\xi \pi.$$

Now we consider the decreasing filtration of the group $\mathcal{E}[X]$ defined by the invariant subgroups

$$\mathfrak{F}_m = \ker\{\mathcal{E}[X] \to \mathcal{E}[X^{(m-1)}]\}, \qquad m \geqq 1,$$

where the homomorphism is induced by the projection map

$$X \to X^{(m-1)}$$

of X into its $(m-1)$th Postnikov system [5] $X^{(m-1)}$. Then we have

THEOREM. *There exists a spectral sequence $E_r^{p,q}$ which converges to the associated graded group of the filtration $\{\mathfrak{F}_m\}$ of the group $\mathcal{E}[X]$*

$$E_\infty^{p,-p} = \mathfrak{F}_p/\mathfrak{F}_{p+1}.$$

The initial terms of the spectral sequence are given by

$$E_1^{p,-p-1} = H^{p-1}(X, \pi_p(X)),$$

$$E_1^{p,-p+1} = H^{p+1}(X^{(p-1)}, \pi_p(X))/\mathrm{Aut}\,\pi_p(X),$$

and $E_1^{p,-p}$ *is given by the extension*

$$1 \to H^p(X^{(p-1)}, \pi_p(X)) \to E_1^{p,-p} \to \mathrm{Aut}_{\xi^{p-1}}\pi_p(X) \to 1.$$

The other terms are zero

$$E_r^{p,q} = 0 \qquad \text{for } p + q \neq -1, 0, +1,$$

where $X^{(p)}$ *is the pth Postnikov system of* X *and*

$$\xi^{p-1} \in H^{p+1}(X^{(p-1)}, \pi_I(X))$$

the $(p-1)$*th Postnikov invariant of* X.

We shall not give the proof here. However, it is better to remark that in this spectral sequence, none except the terms $E_r^{p,-p-1}$ are abelian groups (in fact, they are subgroups of $H^{p-1}(X, \pi_p(X))$). The essential terms

$$E_r^{p,-p}$$

which converge to the group $\mathcal{E}[X]$ are noncommutative groups. And the term

$$E_r^{p,-p+1}$$

is the orbit space of a canonical operation for each $r \geqq 2$

$$E_{r-1}^{p,-p} \times E_{r-1}^{p,-p+1} \to E_{r-1}^{p,-p+1},$$

with distinguished element

$$\xi_{r-1}^{p-1} \in E_{r-1}^{p,-p+1}$$

such that

$$Z_r^{p,-p} \subseteqq E_{r-1}^{p,-p}$$

is the subgroup of stabilizers of ξ_{r-1}^{p-1}. When $r = 2$, the element

$$\xi_1^{p-1} \in E_1^{p,-p+1} = H^{p+1}(X^{(p-1)}, \pi_p(X))/\mathrm{Aut}\,\pi_p(X)$$

is just the orbit of the $(p-1)$th Postnikov invariant ξ^{p-1} of X. The construction and properties of such a spectral sequence do not involve difficulties in the noncommutative case [2]. When the orbit space and the bounded exact sequence (i.e., finite exact sequence without condition on the two ends) are introduced, simple modification of the classical theory of spectral sequence [3] is needed. This is nothing but a careful verification of some easy conditions.

The terms of the spectral sequence involve the cohomology groups of the Postnikov system of X. However, these are only the lower dimensional groups with respect to the system

$$H^i(X^{(p)}), \qquad i \leqq p+2.$$

Hence, if the space X is supposed to be simply connected, they can be computed from the cohomology of X [7]. In fact, we have

COROLLARY 1. *If X is a simply connected* CW-*complex, then the initial terms of the spectral sequence $E_r^{p,q}$, which converges to the associated graded group of the filtration of the group $\mathcal{E}[X]$, are the following: The essential term, which converges to $\mathcal{E}[X]$, is given by the extension*

$$1 \to \ker \phi_p \to E_1^{p,-p} \to \mathrm{Aut}_{\xi^{p-1}}\, \pi_p(X) \to 1$$

where ϕ_p is the canonical homomorphism

$$\phi_p\colon H^p(X, \pi_p(X)) \to \mathrm{Hom}(\pi_p(X), \pi_p(X)),$$

and the term $E_1^{p,-p+1}$ is the orbit space of the operation of the group Aut $\pi_p(X)$ *on the extension $\hat{E}_1^{p,-p+1}$*

$$0 \to \mathrm{coker}\, \phi_p \to \hat{E}_1^{p,-p+1} \to \mathrm{Hom}(\hat{\pi}_{p+1}(X); \pi_p(X)) \to 0$$

where $\hat{\pi}_{p+1}(X)$ is a quotient group of $\pi_{p+1}(X)$ by the image of Whitehead group.

In the general case, we may apply the Fadell-Hurewicz theorem [8] to compute them but more complicatedly, so that the known invariants, cohomology and homotopy group of X, determine the group $\mathcal{E}[X]$.

We terminate by giving the special case when X has only two non-vanishing homotopy groups with invariant

$$\xi \in H^{m+1}(A, n; B).$$

Then it is evident that the spectral sequence is trivial, so we obtain the group $\mathcal{E}[X]$ as a double extension

$$1 \to H^m(A, n; B) \to E \to \mathrm{Aut}_\xi B \to 1,$$
$$1 \to E \to \mathcal{E}[X] \to \mathrm{Aut}\, A_\xi \to 1$$

where the subgroup of Aut A, Aut A_ξ, is given by $\alpha \in \mathrm{Aut}_\xi$ if and only if $\theta(\xi, \beta) = \alpha^*(\xi)$ for some $\beta \in \mathrm{Aut}\, B$. Here we denote by α^* the unique automorphism induced by α

$$\alpha^*: H^*(A, n; B) \to H^*(A, n; B),$$

and it is easy to see that the image and kernel of the canonical homomorphism

$$f: \mathcal{E}[X] \to \mathrm{Aut}\, A \oplus \mathrm{Aut}\, B$$

are given by

$$\mathrm{Im}\, f = \mathrm{Aut}\, A_\xi \oplus \mathrm{Aut}_\xi B,$$
$$\ker f = H^m(A, n; B).$$

Hence we have

COROLLARY 2. *If X is a space of two nonvanishing homotopy groups with invariant $\xi \in H^{m+1}(A, n; B)$, then the group $\mathcal{E}[X]$ is given by the extension*

$$1 \to H^m(A, n; B) \to \mathcal{E}[X] \to \mathrm{Aut}\, A_\xi \oplus \mathrm{Aut}_\xi B \to 1.$$

If A and B are finitely generated abelian groups, then $\mathcal{E}[X]$ is also a finitely generated group.

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