INEQUALITIES FOR GENERAL MATRIX FUNCTIONS

BY MARVIN MARCUS[1] AND HENRYK MINC[2]

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I. **Introduction.** In [4] Schur proved the following beautiful result. If H is a subgroup of the symmetric group of degree m, S_m, and $\chi(\sigma)$ is a character of degree 1 of H, then

$$\det A \leqq \sum_{\sigma \in H} \chi(\sigma) \prod_{t=1}^{m} a_{t\sigma(t)} \tag{1}$$

for any m-square positive semi-definite hermitian matrix A. Observe that if H is the identity group, the inequality (1) is the Hadamard determinant theorem $\det A \leqq \prod_{i=1}^{m} a_{ii}$. In [3] it was conjectured that per $A \geqq \prod_{i=1}^{m} a_{ii}$ and in [2] this inequality was proved. Here per $A = \sum_{\sigma \in S_m} \prod_{t=1}^{m} a_{t\sigma(t)}$ is the permanent of A.

The purpose of the present paper is to announce some inequalities for the general matrix function

$$d_\chi(A) = \sum_{\sigma \in H} \chi(\sigma) \prod_{t=1}^{m} a_{t\sigma(t)}. \tag{2}$$

We shall see subsequently that Schur's inequality (1) is an immediate corollary to our Theorem 4.

II. **Main results.**

THEOREM 1. *If N is m-square normal with characteristic roots $\eta_1, \cdots, \eta_m$, then*

$$\left| d_\chi(N) \right| \leqq \frac{1}{m} \sum_{i=1}^{m} \left| \eta_i \right|^m. \tag{3}$$

In case $\chi \equiv 1$, we have the following generalization of the van der Waerden conjecture in the non-negative hermitian case [3; 5].

THEOREM 2. *Let A be an m-square positive semi-definite hermitian. Let the ith row sum of A be denoted by r_i, $i = 1, \cdots, m$, and suppose $\sum_{i=1}^{m} r_i = r > 0$. Then*

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(4) $$d_1(A) \geqq h \prod_{i=1}^{m} |r_i|^2/r^m$$

where h is the order of the group H. Equality holds in (4) *if and only if either* (i) *A has a zero row, or* (ii) $\rho(A)=1$, *where* $\rho(A)$ *is the rank of A.*

Let $Q_{m,n}$ be the totality of strictly increasing sequences ω of length m chosen from $1, \cdots, n$, $1 \leqq \omega_1 \leqq \cdots \leqq \omega_m \leqq n$. If ω and τ are any two sequences of length m chosen from $1, \cdots, n$, then $A[\omega|\tau]$ is the m-square matrix whose (i, j) entry is a_{ω_i,τ_j}, $i=1, \cdots, m$, $j=1, \cdots, m$. In case $\omega, \tau \in Q_{m,n}$ then $A[\omega|\tau]$ is an m-square submatrix of A.

THEOREM 3. *Let A be an n-square positive semi-definite hermitian matrix with characteristic roots* $\alpha_1 \geqq \alpha_2 \geqq \cdots \geqq \alpha_n$. *Then*

(5) $$\prod_{i=1}^{m} \alpha_{n-i+1} \leqq d_\chi(A[\omega|\omega]) \leqq \frac{1}{m} \sum_{i=1}^{m} \alpha_i^m .$$

THEOREM 4. *If A is* $m \times n$ *and B is* $n \times m$, *then*

(6) $$|d_\chi(AB)|^2 \leqq d_\chi(AA^*)d_\chi(B^*B).$$

In case $\chi \equiv 1$ *equality holds in* (6) *only if* (i) *A has a zero row, or* (ii) *B has a zero column, or* (iii) $A=DPB^*$ *where D is a diagonal matrix and P is a permutation matrix.*

This result without any discussion of equality is found in [3]. Schur's result can now be stated.

COROLLARY 1. *If A is an m-square positive semi-definite hermitian, then*

(7) $$\det A \leqq d_\chi(A).$$

This inequality is easily proved from (6) as follows. Let $A=X^*X$ where X is an m-square triangular matrix. Then

$$\begin{aligned} \det A &= \det X^*X = \det X \det X^* \\ &= d_\chi(X)d_\chi(X^*) = d_\chi(IX)d_\chi(X^*I) \\ &\leqq (d_\chi(X^*X))^{1/2}(d_\chi(X^*X))^{1/2} = d_\chi(A). \end{aligned}$$

Let $\Gamma_{m,n}$ denote the set of n^m sequences $\omega=(\omega_1, \cdots, \omega_m)$, $1 \leqq \omega_i \leqq n$, $i=1, \cdots, n$, and define an equivalence relation in $\Gamma_{m,n}$ by $\omega \sim \tau$ if and only if there exists a $\sigma \in H$ such that $\omega^\sigma=(\omega_{\sigma(1)}, \cdots, \omega_{\sigma(m)}) = (\tau_1, \cdots, \tau_m)=\tau$. For $\omega \in \Gamma_{m,n}$ let $\nu(\omega)$ be the number of $\sigma \in H$ for which $\omega^\sigma=\omega$. By Δ we shall denote a fixed system of distinct representatives for the equivalence relation. For example, if $H=S_m$ we can choose Δ to be the set of $C_{m,n+m-1}$ nondecreasing sequences γ, $\gamma_1 \leqq \cdots \leqq \gamma_m$.

The following result underlies Theorem 1 and is of some interest in itself.

THEOREM 5 (GENERALIZED CAUCHY-BINET EXPANSION). *Let A be $m \times n$ and B be $n \times m$ matrices. Then*

$$(8) \quad d_\chi(AB) = \sum_{\gamma \in \Delta} \frac{1}{\nu(\gamma)} d_\chi(A[1, \cdots, m \mid \gamma]) d_\chi(B[\gamma \mid 1, \cdots, m]).$$

It is well known that certain relations must obtain between subdeterminants of a matrix (the quadratic relations). It is a useful fact that in the case of a unitary matrix a related result is true for the general function d_χ. For each $t = 1, \cdots, n$ and $\gamma \in \Gamma_{m,n}$ let $m_t(\gamma)$ denote the multiplicity of occurrence of the integer t in γ.

THEOREM 6. *If $m = n$ and U is an n-square unitary matrix, then for each $t = 1, \cdots, n$*

$$(9) \qquad \sum_{\gamma \in \Delta} \frac{m_t(\gamma)}{\nu(\gamma)} \mid d_\chi(U[\gamma \mid 1, \cdots, n]) \mid^2 = 1.$$

A matrix is called *doubly stochastic* if every row and column sum is 1. Generalizing what is currently known about the van der Waerden conjecture we have as an immediate consequence of Theorem 2:

COROLLARY 2. *Let A be an m-square doubly stochastic positive semi-definite hermitian matrix. Then, if h is the order of H,*

$$(10) \qquad d_1(A) \geqq \frac{h}{m^m} \cdot$$

Equality holds in (10) *if and only if $A = J_m$, the matrix all of whose entries are $1/m$.*

To see this, simply set each $r_i = 1$ and $r = m$ in (4). The equality can hold if and only if every row of A is a multiple of the first row. Since each row sum is 1 it follows that all the rows are identical, say $(a_{11}, \cdots, a_{1m})$. Since the jth column sum is 1 it follows that $a_{1j} = 1/m$ and hence $A = J_m$.

COROLLARY 3. *If A is an m-square matrix with singular values $\alpha_1 \geqq \cdots \geqq \alpha_m$, then*

$$(11) \qquad \mid d_\chi(A) \mid^2 \leqq \frac{1}{m} \sum_{i=1}^{m} \alpha_i^{2m}.$$

In case $\chi \equiv 1$ *the equality holds in* (11) *if and only if* $A = DP$ *where* D *is a diagonal matrix each of whose main diagonal entries have the same absolute value and* P *is a permutation matrix corresponding to a permutation in* H.

This follows directly from (6) and (3).

COROLLARY 4. *Let* N *be an* m*-square normal matrix and let* $A = NN^* = N^*N$. *Let* $A^{1/2}$ *denote the unique positive semi-definite determination of the square root of* A. *Then*

$$|d_\chi(N)| \leqq d_\chi(A^{1/2}). \tag{12}$$

If $\chi \equiv 1$ *and* (12) *is equality, then* N *is of the form* $DPA^{1/2}$ *where* D *is a diagonal matrix and* P *is a permutation matrix.*

COROLLARY 5. *If* N *is an* m*-square doubly stochastic, normal and has non-negative entries, then*

$$\text{per } N \leqq \frac{\rho(N)}{m} \cdot \tag{13}$$

The inequality is strict unless either N *is a permutation matrix or* $m = 2$ *and* $N = J_2$.

The characteristic roots of N do not exceed 1 in modulus and exactly $\rho(N)$ of them are nonzero. Thus (13) follows immediately from (3). Now suppose (13) is equality. Then every nonzero characteristic root of N is of modulus 1. By the Perron-Frobenius theorem [1] we obtain P and Q, permutation matrices, such that PNQ is a direct sum of primitive matrices [1]. The moduli of the characteristic roots of N and PNQ are the same and $\rho(N) = \rho(PNQ)$. Thus each of the primitive main diagonal blocks in PNQ has precisely one characteristic root equal 1, the rest 0. Thus PNQ is a direct sum of matrices J_{m_i}, $i = 1, \cdots, r$, $m_i \geqq 2$, $i = 1, \cdots, r$, together with an h-square identity matrix: $r = \rho(N) - h$, $\rho(N) \geqq h \geqq 0$. Suppose $r > 0$. Then

$$\text{per } N = \text{per } PNQ = \prod_{i=1}^{r} \frac{m_i!}{m_i^{m_i}} = \frac{\rho(N)}{m} = \frac{h + r}{h + \sum_{i=1}^{r} m_i} \geqq \frac{r}{\sum_{i=1}^{r} m_i} \geqq \frac{r}{\prod_{i=1}^{r} m_i} \cdot$$

Hence

$$\prod_{i=1}^{r} m_i! \geqq r \prod_{i=1}^{r} m_i^{m_i - 1}.$$

This implies $r = 1$, $m_1 = 2$ and $PNQ = I_{m-2} \dotplus J_2$. But then per $PNQ = \frac{1}{2}$

while $\rho(PNQ)/m=(m-1)/m$. Thus $m=2$. If $r=0$, N is clearly a permutation matrix.

COROLLARY 6. *If A is an m-square doubly stochastic matrix with nonnegative entries, then*

$$\text{per } A \leqq \left(\frac{\rho(A)}{m}\right)^{1/2}. \tag{14}$$

Equality holds in (14) *if and only if A is a permutation matrix.*

This follows from (13) and the fact that per $A \leqq (\text{per}(A^*A))^{1/2}$ [3].

III. **Method of proof.** Let V be an n-dimensional unitary space with inner product (x, y) and let $V^{(m)}$ be the tensor product of V with itself m times. For $x_i \in V$, $i=1, \cdots, m$, define the symmetry operator

$$T(x_1 \otimes \cdots \otimes x_m) = \sum_{\sigma\in H} \chi(\sigma) x_{\sigma^{-1}(1)} \otimes \cdots \otimes x_{\sigma^{-1}(m)}.$$

Then $T^2=hT$, $T^*=T$ where h is the order of the subgroup H. Set $x_1 * \cdots * x_m = T(x_1 \otimes \cdots \otimes x_m)$ and observe that

$$(x_1 * \cdots * x_m, y_1 * \cdots * y_m) = hd_\chi(A) \tag{15}$$

where $a_{ij} = (x_i, y_j)$. The inner product in (15) is the standard one in $V^{(m)}$, $(x_1 \otimes \cdots \otimes x_m, y_1 \otimes \cdots \otimes y_m) = \prod_{i=1}^m (x_i, y_i)$. It turns out that Theorem 5 is a restatement of Parseval's Theorem in the symmetry class of tensors $T(V^{(m)})$. We then apply (8) to a matrix of the form U^*DU where D is diagonal and U is unitary to obtain Theorem 6. We can prove the inequality (3) as follows. Let $N = U^* \operatorname{diag}(\eta_1, \cdots, \eta_m) U$ and set $c_\gamma = |d_\chi(U[\gamma|1, \cdots, m])|^2$ for $\gamma \in \Delta$. Then from Theorem 5

$$\begin{aligned}
|d_\chi(N)| &= \left|\sum_{\gamma\in\Delta} \frac{c_\gamma}{\nu(\gamma)} \prod_{t=1}^m \eta_t^{m_t(\gamma)}\right| \leqq \sum_{\gamma\in\Delta} \frac{c_\gamma}{\nu(\gamma)} \prod_{t=1}^m |\eta_t|^{m_t(\gamma)} \\
&\leqq \sum_{\gamma\in\Delta} \frac{c_\gamma}{\nu(\gamma)} \left(\frac{\sum_{t=1}^m m_t(\gamma)|\eta_t|}{m}\right)^m \\
&\leqq \sum_{\gamma\in\Delta} \frac{c_\gamma}{\nu(\gamma)} \frac{1}{m} \sum_{t=1}^m m_t(\gamma) |\eta_t|^m \\
&= \frac{1}{m}\sum_{t=1}^m |\eta_t|^m \sum_{\gamma\in\Delta} \frac{m_t(\gamma)c_\gamma}{\nu(\gamma)} = \frac{1}{m}\sum_{t=1}^m |\eta_t|^m.
\end{aligned}$$

The remaining results can be proved by similar techniques. Thus inequality (4) is obtained by projecting a decomposable element of the symmetry class $T(V^{(m)})$ onto a suitable tensor and using the Cauchy-Schwarz inequality.

The discussion of the cases of equality requires special and somewhat involved arguments.

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University of California, Santa Barbara