ON THE LIFTING PROPERTY. III[1]

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1. Let Z be a locally compact space and $\mu\neq 0$ a positive Radon measure on Z. Let $M_R^\infty(Z, \mu)$ be the Banach algebra of all bounded real-valued μ-measurable functions defined on Z, endowed with the norm $f\to\|f\|_\infty=\sup_{z\in Z}|f(z)|$. Let $C_R^\infty(Z)$ be the subalgebra of $M_R^\infty(Z,\mu)$ consisting of all bounded continuous functions on Z and $\mathcal{K}(Z)$ the subalgebra of all $f\in C_R^\infty(Z)$ having compact support. For two functions f and g defined on Z we shall write $f\equiv g$ whenever f and g coincide locally almost everywhere.

Let now $T:f\to T_f$ be a mapping of $M_R^\infty(Z, \mu)$ into $M_R^\infty(Z, \mu)$. Properties of T such as those listed below will be considered in what follows:

(I) $T_f\equiv f$;
(II) $f\equiv g$ implies $T_f=T_g$;
(III) $T_1=1$;
(IV) $f\geqq 0$ implies $T_f\geqq 0$;
(V) $T_{\alpha f+\beta g}=\alpha T_f+\beta T_g$;
(VI) $T_{fg}=T_fT_g$;
(VII) $T_f=f$ if $f\in C_R^\infty(Z)$.

A mapping $T:f\to T_f$ of $M_R^\infty(Z, \mu)$ into $M_R^\infty(Z,\mu)$ satisfying (I)–(V) will be called a *linear lifting* of $M_R^\infty(Z, \mu)$; if the condition (VI) is also verified the mapping will be called a *lifting* of $M_R^\infty(Z,\mu)$. A *strong linear lifting* [*strong lifting*] of $M_R^\infty(Z, \mu)$ is a linear lifting [lifting] which verifies also (VII).

If $T:f\to T_f$ is a lifting of $M_R^\infty(Z, \mu)$ and A is a μ-measurable set then we shall denote by $\rho_T(A)$ the set defined by the equation[2] $T_{\phi_A}=\phi_{\rho_T(A)}$. For each $z\in Z$ denote by $\mathcal{V}_T(z)$ the set of all parts $\rho_T(V)$ where V belongs to[3] $\mathcal{V}(z)$ and is μ-measurable. If τ_T is the set of all parts $\rho_T(A)-N$ where A is μ-measurable and N is locally μ-negligible then τ_T is a topology on Z (this result is essentially due to J. Oxtoby and has been given by him in a lecture at Yale in the fall of 1960).

THEOREM 1. *Let $T:f\to T_f$ be a lifting of $M_R^\infty(Z,\mu)$. Then the following assertions are equivalent*: (1.1) *T is a strong lifting*; (1.2) *There is*

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[2] For $X\subset Z$, ϕ_X denotes the characteristic function of X.

[3] For each $z\in Z$, $\mathcal{V}(z)$ denotes the set of all neighborhoods of z.

$V \subset M_R^\infty(Z, \mu)$, *dense in* $\mathcal{K}(Z)$ *for the topology induced by the norm* $f \to \|f\|_\infty$, *such that* $T_f = f$ *for all* $f \in V$; (1.3) $\rho_T(U) \supset U$ *for all* $U \subset Z$ *open*; (1.4) $\rho_T(F) \subset F$ *for all* $F \subset Z$ *closed*; (1.5) τ_T *is stronger than the topology of* Z; (1.6) *For each* $z \in Z$, $\mathcal{V}_T(z)$ *is a fundamental system of neighborhoods of* z.

Let $M_R(Z, \mu)$ be the algebra of all locally bounded (that is bounded on every compact) real-valued μ-measurable functions defined on Z. By $C_R(Z)$ we shall denote the subalgebra or all real-valued continuous functions on Z. A mapping $T: f \to T_f$ of $M_R(Z, \mu)$ into $M_R(Z, \mu)$ which verifies conditions (I)–(VI) will be called a *lifting* of $M_R(Z, \mu)$. A *strong lifting* of $M_R(Z, \mu)$ is a lifting $T: f \to T_f$ such that $T_f = f$ for all $f \in C_R(Z)$.

THEOREM 2. *The following conditions are equivalent*: (2.1) *There is a strong linear lifting of* $M_R^\infty(Z, \mu)$; (2.2) *There is a strong lifting of* $M_R^\infty(Z, \mu)$; (2.3) *There is a strong lifting of* $M_R(Z, \mu)$.

REMARK. There is a compact space $Z \neq \emptyset$, a positive Radon measure μ on Z with Supp $\mu = Z$ and a lifting $T: f \to T_f$ of $M_R^\infty(Z, \mu)$ such that

$$Z = \bigcup_{f \in C_R(Z)} \{z \mid T_f(z) \neq f(z)\}.$$

2. For a locally compact space B and a positive Radon measure α on B we shall denote by $\mathcal{C}(B, \alpha)$ the set of all locally countable families $(K_j)_{j \in J}$ of disjoint compact parts of B such that the complement of $\bigcup_{j \in J} K_j$ is locally α-negligible.

Let now Z and B be two locally compact spaces and α a positive Radon measure on B. For each mapping $\lambda: b \to \lambda_b$ of B into[4] $\mathfrak{M}_+(Z)$ and each $g \in \mathcal{K}(Z)$ we shall denote by $\langle g, \lambda \rangle$ the mapping $b \to \langle g, \lambda_b \rangle$. Suppose now $\alpha \neq 0$ and let $T: f \to T_f$ be a lifting of $M_R^\infty(B, \alpha)$; to shorten the notation we shall sometimes write $\rho_T(f)$ instead of T_f, for $f \in M_R^\infty(B, \alpha)$. If $\lambda: b \to \lambda_b$ is a mapping of B into $\mathfrak{M}_+(Z)$ we shall write

$$\rho_T[\lambda] = \lambda$$

whenever there is $(K_j)_{j \in J} \in \mathcal{C}(B, \alpha)$ with the following properties:

(1) $\phi_{K_j} \langle g, \lambda \rangle \in M_R^\infty(B, \alpha)$ for every $j \in J$ and $g \in \mathcal{K}(Z)$;

(2) $\rho_T(\phi_{K_j} \langle g, \lambda \rangle) = \phi_{\rho_T(K_j)} \langle g, \lambda \rangle$ for every $j \in J$ and $g \in \mathcal{K}(Z)$.

A mapping $\lambda: b \to \lambda_b$ of B into $\mathfrak{M}_+(Z)$ will be called *appropriate with respect to* (α, T) if:

[4] $\mathfrak{M}(Z)$ is the vector space of all Radon measures on Z and $\mathfrak{M}_+(Z)$ the cone of all positive Radon measures on Z.

(3) $\rho_T[\lambda]=\lambda$;

(4) $\langle g, \lambda\rangle$ is essentially α-integrable for each $g\in\mathcal{K}(Z)$ (that is, λ is scalarly essentially α-integrable if $\mathfrak{M}(Z)$ is endowed with the topology $\sigma(\mathfrak{M}(Z), \mathcal{K}(Z))$).

Let Z and B be two locally compact spaces and μ a positive Radon measure on Z. Recall that a mapping $p: Z\to B$ is μ-proper if it is μ-measurable and if $f\circ p$ is essentially μ-integrable for each $f\in\mathcal{K}(B)$. The Radon measure $p(\mu)$ is then defined by the equations $\langle f, p(\mu)\rangle = \int_B f\circ p\,d\mu$, $f\in\mathcal{K}(B)$. If α is a positive Radon measure on B and $\psi: B\to R$ a locally α-integrable function then $\psi\cdot\alpha$ is the Radon measure defined by the equations $\langle f, \psi\cdot\alpha\rangle=\int_B f\psi\,d\alpha$, $f\in\mathcal{K}(B)$. For the above and for other definitions concerning integration see [1].

THEOREM 3. *Let Z and B be two locally compact spaces, μ a positive Radon measure on Z, p a μ-proper mapping of Z into B and $\nu=p(\mu)$. Let now α be a positive Radon measure on B such that $\nu=\psi\cdot\alpha$ for some locally α-integrable function ψ. Suppose $\alpha\neq 0$ and let $T: f\to T_f$ be a lifting of $M_R^\infty(B, \alpha)$. Then*:

(3.1) *There is a mapping $\lambda: b\to\lambda_b$ of B into $\mathfrak{M}_+(Z)$, appropriate with respect to (α, T) such that*:

(5) $\|\lambda_b\|=\psi(b)$ *locally almost everywhere for α*;

(6) $\int_Z (f\circ p)g\,d\mu=\int_B f(b)\langle g, \lambda_b\rangle d\alpha(b)$ *for every $f\in\mathcal{K}(B)$ and $g\in\mathcal{K}(Z)$*.[5]

(3.2) *Moreover, if T is a strong lifting then λ_b is concentrated on $p^{-1}(\{b\})$ locally almost everywhere for α.*

(3.3) *Let $\lambda': b\to\lambda_b'$ and $\lambda'': b\to\lambda_b''$ be two mappings of B into $\mathfrak{M}_+(Z)$, appropriate with respect to (α, T)*[6] *and such that*:

(7) *λ_b' and λ_b'' are concentrated on $p^{-1}(\{b\})$ locally almost everywhere for α*;

(8) $\mu=\int_B\lambda_b'\,d\alpha(b)=\int_B\lambda_b''\,d\alpha(b)$.

Then $\lambda_b' =\lambda_b'$ locally almost everywhere for α.

The next result is in a certain sense converse to Theorem 3:

THEOREM 4. *Let B be a locally compact space, $\alpha\neq 0$ a positive Radon measure on B with* Supp $\alpha=B$ *and $T: f\to T_f$ a lifting of $M_R^\infty(B, \alpha)$. Then the assertions* (4.1) *and* (4.2) *below are equivalent*:

(4.1) *There is a locally α-negligible set $B_\infty\subset B$ such that $T_f(b)=f(b)$ for each $f\in C_R^\infty(B)$ and $b\notin B_\infty$*;

(4.2) *For every locally compact space Z, positive Radon measure μ on Z and μ-proper mapping p of Z into B such that $\nu=p(\mu)$ is absolutely continuous with respect to α, there is a mapping $\lambda: b\to\lambda_b$ of B*

[5] From (6) we deduce that $\mu=\int_B \lambda_b d\alpha(b)$ (i.e. μ is the integral of λ with respect to α).

[6] Here the lifting T is not necessarily supposed to be strong.

into $\mathfrak{M}_+(Z)$, *appropriate with respect to* (α, T) *and having the properties*:

(9) $\mu = \int_B \lambda_b d\alpha(b)$;

(10) λ_b *is concentrated on* $p^{-1}(\{b\})$ *locally almost everywhere for* α.

REMARK. If T is a lifting having the property stated in (4.1) then there is a strong lifting T' of $M_R^\infty(B, \alpha)$ such that $T'_f(b) = T_f(b)$ for all $f \in M_R^\infty(B, \alpha)$ and $b \notin B_\infty$.

3. Let Z be a locally compact space and μ a positive Radon measure on Z. To simplify some of the following statements we shall say that (Z, μ) has the *strong lifting property* whenever there is a strong lifting of $M_R^\infty(Z, \mu)$.

In the statements below $Z \neq \emptyset$ is a locally compact space and μ a positive Radon measure on Z with Supp $\mu = Z$.

(A) The couple (Z, μ) has the strong lifting property in each of the following cases: (i) Z is metrizable; (ii) (Z, μ) is hyperstonean (that is Z is stonean and every rare set is locally μ-negligible); (iii) μ is atomic.

(B) If (Z, μ) has the strong lifting property and $K \subset Z$, $K \neq \emptyset$ is a compact such that Supp $\mu_K = K$ then (K, μ_K) has the strong lifting property.

(C) If $(K_j)_{j \in J} \in \mathcal{C}(Z, \mu)$ is such that (K_j, μ_{K_j}) has the strong lifting property for each $j \in J$ then (Z, μ) has the strong lifting property.

(D) Let Z_1, Z_2 be two locally compact spaces and μ_1, $\mu_2 \neq 0$ two positive Radon measures on Z_1, Z_2 respectively. Suppose that (Z_1, μ_1) has the strong lifting property, Z_2 is metrizable and Supp $\mu_2 = Z_2$. Then $(Z_1 \times Z_2, \mu_1 \otimes \mu_2)$ has the strong lifting property.

(E) Let $(Z_j)_{j \in J}$ be a family of metrizable compact spaces and for each $j \in J$ let μ_j be a positive Radon measure on Z_j with $\mu_j(Z_j) = 1$ and Supp $\mu_j = Z_j$; let $Z_\infty = \prod_{j \in J} Z_j$ and $\mu_\infty = \otimes_{j \in J} \mu_j$. Then the couple (Z_∞, μ_∞) has the strong lifting property.

Denote by $C_R^\infty(R, +)$ [$C_R^\infty(R, -)$] the algebra of all bounded real-valued functions, defined on the real line R, continuous on the right [continuous on the left]. Let μ be the Lebesgue measure on R.

THEOREM 5. *There is a lifting* $T: f \to T_f$ *of* $M_R^\infty(R, \mu)$ *such that* $T_f = f$ *for every* $f \in C_R^\infty(R, +)$ [$f \in C_R^\infty(R, -)$].

4. REMARKS. (1) Theorem 1 in [1, Chap. 6, pp. 58–63], some of the results in [2] (see also [4]) and the result in [3, §6, pp. 82–84] are particular cases of Theorem 3 above. Theorem 2 in [1, Chap. 6, pp. 64–65] can be also generalized using the strong lifting. For certain other methods and results concerning the disintegration of measures see also [7]. (2) The result in (E) is essentially contained in [6]. (3) It is known that for every (Z, μ), $\mu \neq 0$, there is a lifting

of $M_R^\infty(Z, \mu)$ (see [5] and [6]).

PROBLEM. Decide whether or not every couple (Z, μ), with $\mu \neq 0$ and Supp $\mu = Z$, has the strong lifting property.

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